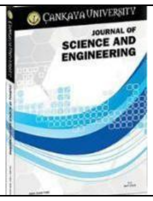
e-ISSN: 2564-7954 CUSJE 18(1): 001-008 (2021) Research Article



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Hybrid Block Method for Direct Integration of First, Second and Third Order IVPs

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Keywords

Consistency,
Convergence,
Zero-Stability.

Abstract

Construction of numerical methods for the solution of initial value problems (IVPs) in ordinary differential equations (ODEs) has been considered overwhelmingly in literature. However, the use of a single numerical method for the integration of ODEs of more than one order has not been commonly reported. In this paper, we focus on the development of a numerical method capable of obtaining the numerical solution of first, second and third-order IVPs. The method is formulated from continuous schemes obtained via collocation and interpolation techniques and applied in a block-by-block manner as a numerical integrator for first, second and third-order ODEs. The convergence properties of the method are discussed via zero-stability and consistency. Numerical examples are included and comparisons are made with existing methods in the literature.

1. Introduction

We propose in this paper, a block method of the form

$$\begin{cases} \sum_{i=0}^{k} \alpha_i(t) y_{n+i} = h\left(\sum_{i=0}^{k} \beta_i(t) f_{n+1} + \beta_j(t) f_{n+j}\right) + h^2\left(\sum_{i=0}^{k} \lambda_i(t) g_{n+i} + \lambda_j(t) g_{n+j}\right) \\ + h^3\left(\sum_{i=0}^{k} \delta_i(t) w_{n+i} + \delta_j(t) w_{n+j}\right) \end{cases} \quad (1)$$

for direct solution of

$$\begin{cases} y'(x) = f(x, y(x)), y(x_0) = y_0 \\ y''(x) = f(x, y(x), y'(x)), y(x_0) = y_0, y'(x_0) = y'_0 \\ y'''(x) = f(x, y(x), y'(x), y''(x)), y(x_0) = y_0, y'(x_0) = y'_0, y''(x_0) = y''_0 \end{cases} \quad (2)$$

where either of $\alpha_0(t)$ and $\beta_0(t)$ do not varnish, $\alpha_k(t) = 1$, $\beta_k(t) \neq 0$ and $k = 1$.

The solution of either $y'''(x) = f(x, y(x), y'(x), y''(x))$ or $y''(x) = f(x, y(x), y'(x))$ has been extensively discussed in the literature using different approaches. Lambert [1, 2], Awoyemi [3] and Brugnano and Trigiante [4], among others, reduced higher-order initial value problems (IVPs) to a system of first-order equations. Resulting from this is the increase in the dimension of the problem which leads to more computation.

Awoyemi ([5], [6]) and Kayode ([7]) successfully applied numerical algorithms as integrators of fourth-order initial value problems directly. However, the implementation in predictor-corrector mode has been reported to be more costly since the subroutines for incorporating the starting values lead to lengthy computational time, see

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Received: May 29, 2020, Accepted: March 14, 2021

(Jator [8]). This setback was addressed by Vigo-Aguiar and Ramos [9], Jator [10], Mohammed [11], Kayode et al. [12], Awoyemi et al. [13], Yap and Ismail [14], Hussain et al. [15], Ismail et al. [16], Ramos et al. [17], Adeyefa [18], among others who independently proposed block methods for solving higher order ordinary differential equation which do not require the development of separate predictors, but simultaneously generate approximations at different grid points within the interval of integration. Of recent, Ogunniran et al. developed Optimized three-step hybrid block method for stiff to integrate stiff IVPs [19] and later considered linear stability analysis of Runge-Kutta Methods for Singular Lane-Emden Equations [20]). Singh et al. developed an efficient optimized adaptive step-size hybrid block method for integrating differential systems [21]. These methods, though efficient in integrating the targeted ODEs but not capable to solve more than one order.

The formulation of block method to integrate IVPs of order one or higher order has been widely reported in the literature but to use a formulated block method for integration of two or more order IVPs, say first, second and third-order ODEs, has not been commonly reported. Thus, the focus of this paper is to formulate a self-starting method for the numerical integration of first, second and third-order IVPs.

In what immediately follows in the next section, we consider the formulation of the proposed block method.

2. Materials and Methods

In this section, we set $i = 0,1,\ j = \frac{3}{4}$ and formulate a new one-step hybrid method capable of solving first, second and third-order ODEs, employing Chebyshev polynomials as our basis function. Thus, we introduce

$$y(x) = \sum_{j=0}^{k+8} a_j T_j(x) \tag{3}$$

Equation (3) is interpolated at $x = x_n$, its first and second derivatives are collocated at $x = x_{n+v}, v = 0, \frac{3}{4}, 1$ while its third derivative is collocated at $x = x_{n+c}, c = 0, \frac{3}{4}$.

As a result, we have

$$\left.\begin{aligned} &\sum_{j=0}^{k+8} a_j T_j(x_n) = y_n \\ &\sum_{j=1}^{k+8} j a_j T_{n+m}^{j-1} = f_{n+m} \\ &\sum_{j=2}^{k+8} j(j-1) a_j T_{n+m}^{j-2} = g_{n+m} \\ &\sum_{j=3}^{k+8} j(j-1)(j-2) a_j T_{n+c}^{j-3} = w_{n+c} \end{aligned}\right\} \tag{4}$$

Solving equation (4) using the Gaussian elimination approach to get the unknown variables $a's$ that are substituted into equation (3). This yields a continuous implicit scheme of the form:

$$y(x) = h\left(\sum_{i=0}^{1} \beta_i(t) f_{n+i} + \beta_{\frac{3}{4}}(t) f_{n+\frac{3}{4}}\right) + h^2\left(\sum_{i=0}^{1} \lambda_i(t) g_{n+i} + \lambda_{\frac{3}{4}}(t) g_{n+\frac{3}{4}}\right) + h^3\left(\delta_0(t) w_0 + \delta_{\frac{3}{4}}(t) w_{n+\frac{3}{4}}\right) \tag{5}$$

where $t = \dfrac{2x - 2x_n - h}{h}$.

Equation (5), when evaluated at $x = x_{n+c_j}, c_j = 1, \frac{3}{4}$ i.e. $t = 1, \frac{1}{2}$ respectively, yields

$$\begin{pmatrix} y_{n+\frac{3}{4}} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} y_n + hD \begin{pmatrix} f_n \\ f_{n+\frac{3}{4}} \\ f_{n+1} \end{pmatrix} + h^2 E \begin{pmatrix} g_n \\ g_{n+\frac{3}{4}} \\ g_{n+1} \end{pmatrix} + h^3 F \begin{pmatrix} w_n \\ w_{n+\frac{3}{4}} \\ w_{n+1} \end{pmatrix} \tag{6}$$

where the values of D, E, and F are

$$D = \begin{pmatrix} \frac{88593}{286720} & \frac{1203}{1120} & -\frac{181521}{286720} \\ \frac{2626}{8505} & \frac{10496}{8505} & -\frac{19}{35} \end{pmatrix}, E = \begin{pmatrix} \frac{747}{20480} & \frac{153}{4480} & \frac{2187}{57344} \\ \frac{59}{1620} & \frac{128}{2835} & \frac{1}{28} \end{pmatrix}, F = \begin{pmatrix} \frac{963}{573440} & \frac{99}{5120} & 0 \\ \frac{19}{11340} & \frac{8}{405} & 0 \end{pmatrix}$$

Equation (6) is our proposed first, second and third-order IVPs solver.

3. Basic Properties of the Method

We shall consider in this section, the analysis of the basic properties of this method such as order, error constant, zero stability and consistency is investigated.

3.1. Order and Error Constant

Equation (6) derived is a discrete scheme belonging to the class of LMMs of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} w_j f_{n+j} + h^2 \sum_{j=0}^{k} \beta_j g_{n+j} + h^3 \sum_{j=0}^{k} \gamma_j G_{n+j} \tag{7}$$

Following the work of Fatunla [26] and Lambert [2], we define the local truncation error associated with equation (7) by the difference operator

$$L[y(x):h] = \sum_{j=0}^{k} [\alpha_j y(x_n + jh) - hw_j f(x_n + jh) - h^2 \beta_j g(x_n + jh) - h^3 \gamma_j G(x_n + jh)] \tag{8}$$

where y(x) is an arbitrary function, continuously differentiable on [a, b]. Expanding (8) in Taylor series about the point x, we obtain the expression

$$L[y(x);h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \text{K} + C_{p+2} h^{p+2} y^{p+2}(x)$$

where the C_0 , C_1 , C_2 ⋰ C_p ⋰ C_{p+2} are obtained as $C_0 = \sum_{j=0}^{k} \alpha_j$, $C_1 = \sum_{j=1}^{k} j\alpha_j$, $C_2 = \frac{1}{2!} \sum_{j=1}^{k} j^2 \alpha_j$,

$$C_q = \frac{1}{q!} \left[\sum_{j=1}^{k} j^q \alpha_j - q(q-1) \sum_{j=1}^{k} \beta_j j^{q-2} - q(q-1)(q-2) \sum_{j=1}^{k} \gamma_j j^{q-3} \right].$$

In the spirit of Lambert [2], for first, second and third-order methods respectively, equation (8) is of order p if $C_0 = C_1 = C_2 = \text{K}\ C_p = 0$ and $C_{p+r} \neq 0$ for r = 1, 2, 3. The $C_{p+r} \neq 0$ is called the error constant and $C_{p+r} h^{p+r} y^{p+r}(x_n)$ is the principal local truncation error at the point x_n.

Thus, the block (6) is of order p = 6 and error constants

$$C_{p+r} = \left[\frac{-3159}{328833433600}, \frac{382563}{289373421568000} \right]^T.$$

3.2. Zero-Stability of the Method

To analyze the zero-stability of the method, we present (7) in vector notation form of column vectors $e=(e_1 \ldots e_r)^T$, $d=(d_1 \ldots d_r)^T$, $y_m=(y_{n+1} \ldots y_{n+r})^T$, $F(y_m)=(f_{n+1} \ldots f_{n+r})^T$, $G(y_m)=(g_{n+1} \ldots g_{n+r})^T$, $W(y_m)=(w_{n+1} \ldots w_{n+r})^T$ and matrices $A=(a_{ij})$, $B=(b_{ij})$.

Thus, equation (6) forms the block formula

$$A_0 y_m = hBF(y_m) + A^1 y_n + hbf_n + h^2 DG(y_m) + h^2 dg_n + h^3 VW(y_m) + h^3 uT_n \tag{9}$$

where h is a fixed mesh size within a block.

In line with equation (9), $A^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A^1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

The first characteristic polynomial of the block hybrid method is given by

$$\rho(R) = \det(RA^0 - A^1) \tag{10}$$

Substituting $A^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A^1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ in equation (10) and solving for R, the values of R are obtained as 0 and 1.

According to Fatunla in [26, 27], the block formulae represented by equation (6) are zero-stable, since in equation (10), $\rho(R)=0$, satisfy $|R| \leq 1, j=1$ and for those roots with $|R_j|=1$, the multiplicity does not exceed two.

3.3. Consistency and Convergence of the Method

The linear multistep method (7) is said to be consistent if it has order $p \geq 1$. The method is consistent being of order 6.

According to the theorem of Dahlquist in [28], the necessary and sufficient condition for a linear multistep method to be convergent is to be consistent and zero stable. Since the method satisfies the two conditions hence it is convergent.

3.4. Numerical Experiment

We consider here, four test problems which include first, second and third-order ordinary differential equations to test the effectiveness of this new scheme.

Problem 1: We consider the first-order IVP $y' = 0.5(1-y), y(0) = 0.5, h = 0.1$ with the exact solution $y(x) = 1 - 0.5e^{-0.5x}$. This IVP was solved by Ajileye [29] and Sunday [30]. The numerical solution is displayed in Table 1.

Table 1. Comparison of errors of the proposed method and the existing methods (Ajileye [29], Sunday [30]).

T	Error in the new method	Error in [29]	Error in [30]
0.1	0	1.218026×10^{-13}	5.574430×10^{-12}
0.2	0	1.399991×10^{-13}	3.946177×10^{-12}
0.3	0	1.184941×10^{-12}	8.183232×10^{-12}
0.4	1×10^{-10}	1.538991×10^{-12}	3.436118×10^{-15}
0.5	1×10^{-10}	1.110001×10^{-12}	1.929743×10^{-10}
0.6	1×10^{-10}	5.270229×10^{-12}	1.879040×10^{-10}
0.7	0	2.108980×10^{-12}	1.776835×10^{-10}
0.8	0	1.297895×10^{-11}	1.724676×10^{-10}
0.9	0	3.082290×10^{-11}	1.847545×10^{-10}
1.0	0	4.121925×10^{-11}	3.005770×10^{-10}

Problem 2: We consider the IVP $y'' = y', \; y(0) = 0, y'(0) = -1, \; h = 0.1$ with the exact solution, $y(x) = 1 - e^{x}$ which has been solved in Mohammed and Adeniyi [31] with step number k =5. The numerical solution is displayed in Table 2.

Table 2. Comparison of errors of the proposed method and the existing methods (Mohammed and Adeniyi [31], Mohammed [32]).

T	Error in the new method	Error in [31]	Error in [32]
0.1	1×10^{-10}	2.0040×10^{-7}	2.19800×10^{-5}
0.2	1×10^{-10}	5.3860×10^{-7}	6.07040×10^{-6}
0.3	0	8.8400×10^{-7}	1.00510×10^{-5}
0.4	2×10^{-10}	1.2297×10^{-6}	1.40253×10^{-5}
0.5	4×10^{-10}	1.5752×10^{-6}	1.79934×10^{-5}
0.6	4×10^{-10}	1.9204×10^{-6}	2.16162×10^{-5}
0.7	3×10^{-10}	2.5060×10^{-6}	2.99300×10^{-5}
0.8	3×10^{-10}	3.1060×10^{-6}	3.45610×10^{-5}
0.9	3×10^{-10}	3.7050×10^{-6}	4.11140×10^{-5}
1.0	1×10^{-10}	4.3040×10^{-6}	4.76560×10^{-5}

Problem 3: We consider the third-order IVP

$$y''' + y' = 0, \;\; y(0) = 0, y'(0) = 1, y''(0) = 2$$

with the exact solution, $y(x) = 2(1 - \cos x) + \sin x$ which has been solved in Anake et al. [25] with step number k = 5. The numerical solution is displayed in Table 3.

Table 3. Comparison of error of the proposed method and error in Anake et al. [25].

x	Error in the new method	Error in [25]
0.1	3.30E-11	1.61E-09
0.2	9.67E-10	1.04E-08
0.3	3.80E-09	2.96E-08
0.4	9.53E-09	2.32E-07
0.5	1.93E-08	4.54E-07
0.6	3.39E-08	1.48E-06
0.7	5.45E-08	2.87E-06
0.8	8.22E-08	4.68E-06
0.9	1.17E-07	6.92E-06
1.0	1.62E-07	9.60E-06

Problem 4: We consider non-linear IVP

$$y'' - x(y')^2 = 0, \ \ y(0) = 1, y'(0) = \frac{1}{2}, h = 0.003125$$

whose exact solution is

$$y(x) = 1 + \frac{1}{2} In\left(\frac{2+x}{2-x}\right).$$

The numerical solution is displayed in Table 4.

Table 4. Comparing the errors of the new block and existing methods for Problem 4

x	Error in the new method, *k*=1	Error in [24], *k*=3	Error in [23], *k*=6	Error in [22], *k*=6
0.1	0	5.850875E-13	9.577668E-10	0.1329867326E-09
0.2	1.0E-09	2.848832E-12	2.368709E-09	0.5872691257E-08
0.3	1.0E-09	6.328715E-12	3.732243E-09	0.1327845616E-07
0.4	1.0E-09	6.756392E-09	5.475119E-09	0.2317829012E-07
0.5	1.0E-09	1.380119E-08	1.142189E-08	0.3218793564E-07
0.6	1.0E-09	2.174817E-08	4.567944E-08	0.6871246012E-07
0.7	1.0E-09	1.073052E-07	2.055838E-06	0.1012728156E-06
0.8	1.0E-09	2.001340E-07	4.248299E-06	0.1231093271E-06
0.9	2.0E-09	3.088383E-07	6.660458E-06	0.2019286712E-06
1.0	1.0E-09	9.805074E-07	9.445166E-06	0.2990871645E-06

3.5. Discussion of Results

The results obtained from the four test problems considered are summarized in Tables 1-4. In Table 1, we compare the solution of problem 1 using the proposed method with Ajileye et al. (k = 2) and Sunday et al. (k = 5) methods. In Problem 2, our step length is h = 0.1 against h = 0.01 used in Mohammed and Adeniyi [31]. The proposed method still gives better accuracy even with larger h. In Table 3, the method developed by Anake et al. [25] is of step number k = 5. The proposed method is of step number k = 1 and it compares favourably with the existing methods despite their k > 1 methods.

4. Conclusion

A block method has been applied to solve first, second and third-order ordinary differential equations directly without the construction of additional schemes or employing existing predictors for implementation. Numerical experiments performed using this method show that the method is consistent, efficient and accurate. We, therefore, recommend the method for direct integration of first, second and third-order ordinary differential equations.

Declaration of Competing Interest

The authors declare that there is no competing financial interests or personal relationships that influence the work in this paper.

Authorship Contribution Statement

Emmanuel Oluseye ADEYEFA: Conceptualization, Methodology, Writing- Original draft preparation, Visualization, Investigation. Supervision, Validation, Writing- Reviewing and Editing.
Adeyemi Sunday OLAGUNJU: Software, Data curation

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