Journal of the
Nigerian Mathematical Society
Vol. 38, Issue 2, pp. 211-222, 2019

SPECTRAL-BERNSTEIN RESIDUAL METHOD FOR THE SOLUTION OF BOUNDARY VALUE PROBLEM GOVERNING DEFLECTION OF A BEAM VIA MATLAB

ADEYEMI S. OLAGUNJU[1] AND MOSES C. NWONYE[2]

ABSTRACT. This work focuses on the use of Bernstein basis function in the application of Spectral method for numerical solution of differential equations governing deflection of prismatic beam subject to transverse loading. The solution takes the form of linear combination of Bernstein basis functions and spectral coefficients. The entire solution process is automated in MATLAB and a set of illustrative examples are given to demonstrate the effectiveness and accuracy of the technique.

Keywords and phrases: Bernstein polynomials, Prismatic Beam, MATLAB,Spectral method
2010 Mathematical Subject Classification: 65L10, 65L60

1. INTRODUCTION

A beam is a structural element used for bearing loads, it is typically used for resisting vertical loads, shear forces and bending moments. The loads applied to the beam result in reactive forces at the beam's support points. The total effect of all the forces acting on the beam produces the shear forces and bending moments within the beam, that in turn induce internal stresses, strains and deflections of the beam.
This deflection from of a beam is governed by the differential equation of the form:

$$\frac{y''}{[1+(y')^2]^{3/2}} - \frac{T}{EI}y = \frac{wx(L-x)}{2EI}, \quad 0 \leq x \leq L \tag{1}$$

With boundary condition: $y(0) = y(L) = 0$

Received by the editors April 20, 2018; Revised November 22, 2018 ; Accepted: April 27, 2019
www.nigerianmathematicalsociety.org; Journal available online at https://ojs.ictp.it/jnms/
[1]Corresponding author
[2]

Where

$$\begin{aligned} x &= location\ along\ the\ beam\ (in) \\ T &= Tension\ applied\ (lbs) \\ E &= Young's\ modulus\ of\ elasticity\ of\ the\ beam\ (psi) \\ I &= moment\ of\ inertia\ (in^4) \\ w &= uniform\ loading\ intensity\ (lb/in) \\ L &= length\ of\ beam\ (in) \end{aligned}$$

Over the years numerical methods have proven to be reliable and such that provide constructive means of obtaining solution to this equation, for instance Siang-Yu Tsai [11] provided numerical computation for nonlinear beam problems, Thankane and Stys [10] applied finite difference method for beam equation using mathematica to establish results for beam with free end. Gunakala et al. [8] equally published a journal article on a finite element solution of the beam equation via MATLAB.

In the work of Miletic et al.[7] on Euler-Bernoulli beam equation with boundary conditions, its stability is discussed under dissipative finite element method. On the other hand, Vidar Stienstrom [9] established a numerical simulation of the dynamic beam using the SBP-SAT method. This method involves a stable boundary treatment of the dynamic beam equation (DBE) which two different sets of boundary conditions has been conducted using summation-by-part-simultaneous-approximation-team (SBP-SAT)

This work seeks to establish MATLAB implementation of the Spectral-Bernstein method in the solution of equation (1). This method is essentially an implementation of spectral method in the solution of beam problem using Bernstein polynomial as the basis function.

2. Bernstein polynomials

Bernstein polynomial named after Sergie Natanovich Bernstein in 1912 [4], when he used the polynomials in Bernstein form in a constructive proof for the Stone-Weierstrass approximation theorem [**Bernstein 1912**]. As a result of numerous properties of this polynomial, it has over the years been deployed for numerical solution of several equations cutting across different types of differential, Integral and integro-differential equations.

The Bernstein polynomials of degree n on the interval $[a,\ b]$ are defined by

$$B_{i,n}(t) = \frac{1}{(b-a)^n}\binom{n}{i}(t-a)^i(b-t)^{n-i} \tag{2}$$

$$for\ i = 0, 1, \ldots, n$$

$$where \begin{pmatrix} n \\ i \end{pmatrix} = \frac{n!}{n!(n-1)!}$$

It satisfies a 3-term recursive relation:

$$B_{k,n}(t) = (1 - t) B_{k,n-1}(t) + t B_{k-1,n-1}(t)$$

Table below shows Bernstein polynomials of degree 1-5 within the natural interval

Table 1: Properties of Bernstein polynomials as applied in this study are found in [4].

n	Bernstein Polynomials
1	$B_{0,1} = 1 - t,\ B_{1,1} = t$
2	$B_{0,2} = (1-t)^2,\ B_{1,2} = 2t(1-t),\ B_{2,2} = t^2$
3	$B_{0,3} = (1-t)^3,\ B_{1,3} = 3t(1-t)^2\ B_{2,3} = 3t^2(1-t)\ B_{3,3} = t^3$
4	$B_{0,4} = (1-t)^4,\ B_{1,4} = 4t(1-t)^3, B_{2,4} = 6t^2(1-t)^2, B_{3,4} = 4t^3(1-t), B_{4,4} = t^4$
5	$B_{0,5} = (1-t)^5,\ B_{1,5} = 5t(1-t)^4\ B_{2,5} = 10t^2(1-t)^3, B_{3,5} = 10t^3(1-t)^2\ B_{4,5} = 5t^4(1-t) B_{5,5} = t^5$

3. Spectral Method

Spectral methods were developed in a long series paper by S.A Orszag (1969) [5]. This method is normally accomplished either with collocation or a Galerkin or a Tau (τ) approach. Spectral method is closely related with finite element methods (FEM) since they are built on the same idea; the distinct difference between them is that, spectral method has an advantage over the finite element method because spectral methods uses basis functions that are nonzero over the whole domain while finite element make use of local approach since it uses basis functions that are nonzero only on small sub-domains. The original tau (τ) method for ordinary differential equations approximate the unknown function by a truncated Chebyshev expansion. Thus, Spectral methods have minimal error when compared to finite element method it has excellent error properties as describe in numerical analysis of spectral methods by D.Gottlieb *et al.* (1977) [5] and B. Mercier (1989) [1]. J.P.Boyd (2000) [2] discusses Chebyshev and Fourier spectral method. L.N. Trefethen (2000) [6] established spectral methods in MATLAB. Spectral methods fundamentals in single domains was established by C.Canuto *etal* (2006) [3]. However, this work seeks to explore effectiveness of Bernstein polynomials using the spectral method.

SPECTRAL-BERNSTEIN METHOD

The heart of the matter is to state without proof the following: The use of spectral method demands an accomplishing approach which from previous studies can be collocation, Gelerkin or Tau (τ) approach. The unknown function $y(x)$ in equation (1) represents the deflection, this equation is nonlinear and there is no hope to solve it exactly Beam 4 . However, if the deflection is small then $(y')^2$ is negligible compared to 1, thus the equation simplifies to:

$$y'' - \frac{T}{EI}y = \frac{wx(L-x)}{2EI} \tag{2}$$

Subject to boundary condition: $y(0) = y(L) = 0$

We therefore proffer numerical solution to equation (2) since it is a good approximation to (1) having satisfied the above stated condition. This method involves representing the deflection $y(x)$ in equation (2) by the finite expansion:

$$y_n(x) = \sum_{i=0}^{n} c_i \, \phi_i(x) \tag{3}$$

The efficiency of Bernstein polynomial in numerical approximation made it a choice polynomial in a good number of numerical studies, as a result of this, this work is structured to have equation (3) as Bernstein approximate solution of the form;

$$y_n(x,c) = \sum_{i=o}^{n} c_i(x) \, B_{i,n}(x) \tag{4}$$

Where $B_{i,n}(x)$ *and* c_i are Bernstein polynomials and spectral coefficients respectively. For a chosen n, equation (4) is substituted in the beam equation (2) written as;

$$R \Rightarrow y'' - \alpha y - \beta x (L - x) = 0 \tag{5}$$

Where $\alpha = \frac{T}{EI}$ and $\beta = \frac{w}{2EI}$

The implication of equation (5) is that when the exact solution $y(x)$ is substituted into (2), we have the residual equals to zero.

However, when any other function apart from the exact solution is substituted into (2), it gives a residual which is not equal to zero.

$$R \Rightarrow \bar{y}'' - \alpha \bar{y} - \beta x (L - x) \neq 0 \tag{6}$$

$$R(x,c) \Rightarrow \frac{d}{dx^2}\left(\sum_{i=0}^{n} c_i \, B_{i,n}(x)\right) - \alpha \left(\sum_{i=0}^{n} c_i \, B_{i,n}(x)\right) - \beta t (L-t) \neq 0 \tag{7}$$

Equation (7) is thereafter set to zero at each of $(n - 1)$ equally spaced points within the interval $(0, L)$. This yield a system of $(n - 1)$ equations.

Treatment of Boundary Conditions

At the boundary points of simply supported beam we have;

$$y(0) = y(L) = 0 \tag{8}$$

Considering properties of Bernstein polynomial is illustrated in [Olagunju Maths Phy] and subjecting equation (4) to (8) we have:

$$\begin{aligned} c_0 &= 0 \\ c_n &= 0 \end{aligned}$$

In conjunction with equation (4), this is a reflection of zero deflection at the boundary points.

The Approximate Solution

The system of $(n-1)$ equations from (7) are solved via the use of algebraic solver to yield numerical values for in (8) is solved via the use of algebraic solver, this yields numerical values for $c_1,\ c_2,\ ...\ c_{n-1}$ which are substituted into (4) to yield approximate solution;

$$B_n(y\,;\,t) = \sum_{i=0}^{n} c_i\, B_{i,n}(t)$$

Absolute error $|\,e_n(t)\,|$ is obtained with $e_n(t) = y(t) - B_n(y;\ t)$, where $y(t)$ is the exact solution of the beam equation and $B_n(y\,;\,t)$ is Spectral-Bernstein approximation of the beam equation.
Maximum error is defined as:

$$Max_E(y;\ [a,\ b]) = E_n(y) = \|\, e_n(t)\,\|_\infty = \max_{a \le t \le b} |\, e_n(t)\,|$$

The Entire Solution Procedure that accepts inputs from user is written in MATLAB for n=5 to n=10

```
%MATLAB CODE FOR BEAM EQUATION n=5 to n=10

syms x a0 a1 a2 a3 a4 a5 a6 a7 a8 a9 a10 a11 a12 A B C D E F
%order=input('enter the order of the differential equation>>');
n=input('enter n>>');
A=input('enter A>>');
B=input('enter B>>');
C=input('enter C>>');
D=input('enter D>>');
intA=input('enter the interval a>>');
intB=input('enter the interval b>>');

a(1)=a0;
a(2)=a1;
a(3)=a2;
a(4)=a3;
a(5)=a4;
a(6)=a5;
a(7)=a6;
a(8)=a7;
a(9)=a8;
```

```
a(10)=a9;
a(11)=a10;

Bn=0;
for k=0:1:n;
 Bn=Bn+a(k+1)*nchoosek(n,k)*(x-intA)^k*((intB-x)^(n-k))/(intA-intB)^n;
end
disp(expand(Bn));

y=Bn;
dy=diff(Bn,x,1);
d2y=diff(Bn,x,2);
bc1=input('enter BC1 as y dy or d2y-1>>');
bc2=input('enter BC2 as y dy or d2y-1>>');
bc1a=subs(bc1,x,intA);
bc2a=subs(bc2,x,intB);
eq=A*d2y+B*dy+C*y+D;
%Collocation points boundary point non inclusive
cpb=linspace(intA,intB,n+1);
a0=0;

if n==5;
    a5=0;
elseif n==6;
    a6=0;
elseif n==7;
    a7=0;
elseif n==8;
    a8=0;
elseif n==9;
    a9=0;
elseif n==10;
    a10=0;
end
a=0;
eq2=subs(eq,x,cpb(2:n));
eq3=subs(eq2);
s2=solve(eq3);
if n==5
 a1=s2.a1;
 a2=s2.a2;
 a3=s2.a3;
 a4=s2.a4;
elseif n==6
 a1=s2.a1;
 a2=s2.a2;
 a3=s2.a3;
 a4=s2.a4;
 a5=s2.a5;
 elseif n==7
 a1=s2.a1;
 a2=s2.a2;
 a3=s2.a3;
```

```
 a4=s2.a4;
 a5=s2.a5;
 a6=s2.a6;
 elseif n==8
 a1=s2.a1;
 a2=s2.a2;
 a3=s2.a3;
 a4=s2.a4;
 a5=s2.a5;
 a6=s2.a6;
 a7=s2.a7;
 elseif n==9
 a1=s2.a1;
 a2=s2.a2;
 a3=s2.a3;
 a4=s2.a4;
 a5=s2.a5;
 a6=s2.a6;
 a7=s2.a7;
 a8=s2.a8;
elseif n==10;
 a1=s2.a1;
 a2=s2.a2;
 a3=s2.a3;
 a4=s2.a4;
 a5=s2.a5;
 a6=s2.a6;
 a7=s2.a7;
 a8=s2.a8;
 a9=s2.a9;
end
yn1=subs(y);
xx=intA:0.1:intB;
app=subs(yn1,x,xx');
format short e
ex=input('enter exact solution>>');
exact=double(subs(ex,x,xx'));
error1_BPI=double(abs(exact-app));
disp('    x             exact      error1_BPI  ')
disp([double(xx') exact error1_BPI])
clear
```

Numerical Experiment

Experiment 1 Given that

$$y'' - 1.13636 \times 10^{-6} + 5.818 \times 10^{-5}t - 1.13636 \times 10^{-6}t^2 = 0$$

$$\alpha = 1.13636 \times 10^{-6} \quad and \quad \beta = 1.13636 \times 10^{-6}$$

$$0 \leq t \leq 50$$

$$y(0) = 0 = y(50)$$

determine the deflection of the beam.
The exact solution is;

$$y(t) = 51.1986t + 855994.3391e^{0.0011t} + 904011.2929e^{-0.0011t} - 1.0t^2 \\ -1760005.6320.$$

Table 2 Table of error for different degree of approximation

t	n=3	n=5	n=7	n=9
0	0.0000 e^{-003}	0.0005 e^{-006}	0.4657 e^{-008}	0.4657 e^{-0010}
5	0.0130 e^{-003}	0.1977 e^{-006}	0.5467 e^{-008}	0.5453 e^{-0010}
10	0.0164 e^{-003}	0.1941 e^{-006}	0.4765 e^{-008}	0.4752 e^{-0010}
15	0.0153 e^{-003}	0.1782 e^{-006}	0.5988 e^{-008}	0.5975 e^{-0010}
20	0.0132 e^{-003}	0.1834 e^{-006}	0.5614 e^{-008}	0.5601 e^{-0010}
25	0.0123 e^{-003}	0.1895 e^{-006}	0.4460 e^{-008}	0.4447 e^{-0010}
30	0.0132 e^{-003}	0.1834 e^{-006}	0.5614 e^{-008}	0.5601 e^{-0010}
35	0.0153 e^{-003}	0.1782 e^{-006}	0.5988 e^{-008}	0.5975 e^{-0010}
40	0.0164 e^{-003}	0.1941 e^{-006}	0.4765 e^{-008}	0.4752 e^{-0010}
45	0.0130 e^{-003}	0.1977 e^{-006}	0.5467 e^{-008}	0.5453 e^{-0010}
50	0.0000 e^{-003}	0.0005 e^{-006}	0.4657 e^{-008}	0.4657 e^{-0010}

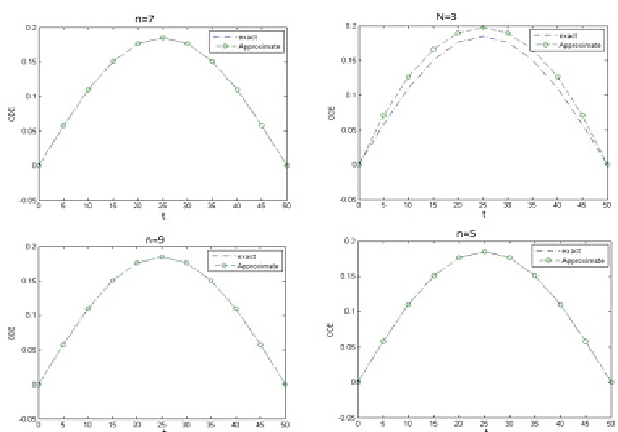

Fig. 1. Bifurcation plot for a variable viscosity when $\Gamma = \Lambda = 1$.

Experiment 2 Determine the deflection of a beam Given that $T = 1000lbs$, $q = 100(lb/in)$, $L = 120in$, $I = 625(in^4)$, $E = 3.0 * 10^7(psi)$ therefore,

$$\alpha = \frac{T}{EI} \quad and \quad \beta = \frac{q}{2EI}$$

$$\alpha = \frac{1000}{3.0 * 10^7 \times 625} \quad and \quad \beta = \frac{100}{2 \times 3.0 * 10^7 \times 625}$$

$$\alpha = 5.3333 \times 10^{-8} \quad and \quad \beta = 2.6667 \times 10^{-9}$$

Hence,

$$y''(t) - 5.3333 \times 10^{-8} y(t) - 2.6667 \times 10^{-9} t(t-120)$$

$$0 \leq t \leq 120$$

$$y(0) = 0 = y(120)$$

.

The exact solution is;

$$y(t) = 6.0001t + 924533.6039e^{0.0002t} + 950513.2716e^{-0.0002t} - 0.05t^2$$
$$-1875046.8755$$

Table 3: Table of error for different degree of approximation

t	n=3	n=5	n=7	n=9
0	0.0000 e^{-003}	0.2561 e^{-008}	0.2561 e^{-009}	0.2561 e^{-0010}
5	0.0005 e^{-003}	0.0130 e^{-008}	0.2584 e^{-009}	0.2584 e^{-0010}
10	0.0009 e^{-003}	0.1681 e^{-008}	0.2251 e^{-009}	0.2251 e^{-0010}
15	0.0011 e^{-003}	0.1797 e^{-008}	0.2496 e^{-009}	0.2496 e^{-0010}
20	0.0012 e^{-003}	0.1686 e^{-008}	0.2548 e^{-009}	0.2548 e^{-0010}
25	0.0013 e^{-003}	0.1496 e^{-008}	0.2538 e^{-009}	0.2538 e^{-0010}
30	0.0013 e^{-003}	0.1250 e^{-008}	0.2599 e^{-009}	0.2599 e^{-0010}
35	0.0012 e^{-003}	0.1077 e^{-008}	0.2672 e^{-009}	0.2672 e^{-0010}
40	0.0011 e^{-003}	0.0976 e^{-008}	0.2765 e^{-009}	0.2765 e^{-0010}
45	0.0011 e^{-003}	0.1287 e^{-008}	0.2512 e^{-009}	0.2512 e^{-0010}
50	0.0010 e^{-003}	0.1525 e^{-008}	0.2359 e^{-009}	0.2359 e^{-0010}
55	0.0010 e^{-003}	0.1773 e^{-008}	0.2181 e^{-009}	0.2181 e^{-0010}
60	0.0010 e^{-003}	0.1524 e^{-008}	0.2457 e^{-009}	0.2457 e^{-0010}
65	0.0010 e^{-003}	0.1307 e^{-008}	0.2647 e^{-009}	0.2647 e^{-0010}
70	0.0010 e^{-003}	0.1525 e^{-008}	0.2359 e^{-009}	0.2359 e^{-0010}
75	0.0011 e^{-003}	0.1287 e^{-008}	0.2512 e^{-009}	0.2512 e^{-0010}
80	0.0011 e^{-003}	0.1209 e^{-008}	0.2532 e^{-009}	0.2532 e^{-0010}
85	0.0012 e^{-003}	0.1077 e^{-008}	0.2672 e^{-009}	0.2672 e^{-0010}
90	0.0013 e^{-003}	0.1483 e^{-008}	0.2366 e^{-009}	0.2366 e^{-0010}
95	0.0013 e^{-003}	0.1729 e^{-008}	0.2305 e^{-009}	0.2305 e^{-0010}
100	0.0012 e^{-003}	0.1686 e^{-008}	0.2548 e^{-009}	0.2548 e^{-0010}
105	0.0011 e^{-003}	0.1797 e^{-008}	0.2496 e^{-009}	0.2496 e^{-0010}
110	0.0009 e^{-003}	0.1449 e^{-008}	0.2484 e^{-009}	0.2484 e^{-0010}
115	0.0005 e^{-003}	0.0130 e^{-008}	0.2584 e^{-009}	0.2584 e^{-0010}
120	0.0000 e^{-003}	0.2328 e^{-008}	0.2328 e^{-009}	0.2328 e^{-0010}

Experiment 3 Determine the deflection of a beam Given that $T = 10000lbs$, $q = 10000(lb/in)$, $L = 120in$, $I = 121(in^4)$, $E = 29 * 10^6(psi)$

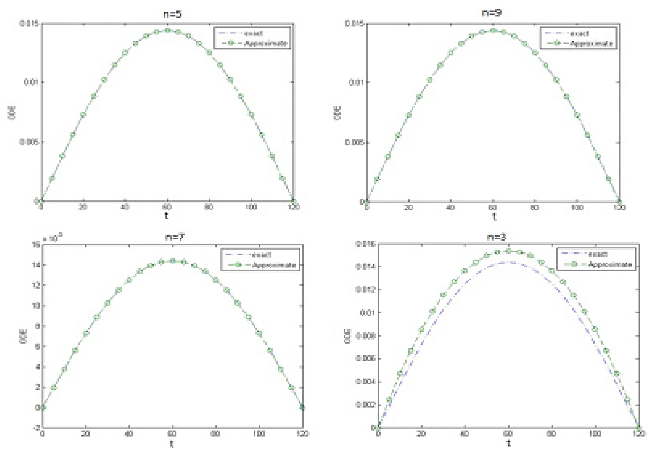

Fig. 2. Bifurcation plot for a variable viscosity when $\Gamma = \Lambda = 1$.

therefore,

$$\alpha = \frac{T}{EI} \quad and \quad \beta = \frac{q}{2EI}$$
$$\alpha = \frac{10000}{29 * 10^6 \times 121} \quad and \quad \beta = \frac{1000}{2 \times 29 * 10^6 \times 121}$$
$$\alpha = 2.850 \times 10^{-6} \quad and \quad \beta = 1.425 \times 10^{-6}$$

Hence,

$$y''(t) - 2.850 \times 10^{-6} y(t) - 1.425 \times 10^{-6} t(120 - t)$$
$$0 \leq t \leq 120$$
$$y(0) = 0 = y(120)$$

.

The exact solution is;

$$y(t) = 0.5t^2 - 157728.6568e^{0.0017t} - 193148.5362e^{-0.0017t}$$
$$-60t + 350877.1930$$

Table 4: Table of error for different degree of approximation

t	n=3	n=5	n=7	n=9
0	0.0000 e^{-001}	0.0000 e^{-003}	0.0006 e^{-007}	0.5821 e^{-0010}
5	0.2867 e^{-001}	0.0771 e^{-003}	0.0951 e^{-007}	0.5545 e^{-0010}
10	0.4802 e^{-001}	0.1116 e^{-003}	0.1199 e^{-007}	0.1877 e^{-0010}
15	0.5991 e^{-001}	0.1218 e^{-003}	0.1195 e^{-007}	0.1938 e^{-0010}
20	0.6603 e^{-001}	0.1200 e^{-003}	0.1143 e^{-007}	0.4657 e^{-0010}
25	0.6788 e^{-001}	0.1143 e^{-003}	0.1111 e^{-007}	0.6165 e^{-0010}
30	0.6680 e^{-001}	0.1090 e^{-003}	0.1105 e^{-007}	0.2436 e^{-0010}
35	0.6393 e^{-001}	0.1061 e^{-003}	0.1120 e^{-007}	0.4833 e^{-0010}
40	0.6026 e^{-001}	0.1058 e^{-003}	0.1130 e^{-007}	0.3575 e^{-0010}
45	0.5657 e^{-001}	0.1075 e^{-003}	0.1136 e^{-007}	0.7176 e^{-0010}
50	0.5350 e^{-001}	0.1098 e^{-003}	0.1131 e^{-007}	0.7376 e^{-0010}
55	0.5148 e^{-001}	0.1118 e^{-003}	0.1121 e^{-007}	0.5224 e^{-0010}
60	0.5077 e^{-001}	0.1126 e^{-003}	0.1114 e^{-007}	0.2013 e^{-0010}
65	0.5148 e^{-001}	0.1118 e^{-003}	0.1121 e^{-007}	0.5224 e^{-0010}
70	0.5350 e^{-001}	0.1098 e^{-003}	0.1131 e^{-007}	0.7377 e^{-0010}
75	0.5657 e^{-001}	0.1075 e^{-003}	0.1130 e^{-007}	0.1355 e^{-0010}
80	0.6026 e^{-001}	0.1058 e^{-003}	0.1130 e^{-007}	0.3575 e^{-0010}
85	0.6393 e^{-001}	0.1061 e^{-003}	0.1120 e^{-007}	0.4834 e^{-0010}
90	0.6680 e^{-001}	0.1090 e^{-003}	0.1105 e^{-007}	0.2436 e^{-0010}
95	0.6788 e^{-001}	0.1143 e^{-003}	0.1111 e^{-007}	0.6165 e^{-0010}
100	0.6603 e^{-001}	0.1200 e^{-003}	0.1143 e^{-007}	0.4658 e^{-0010}
105	0.5991 e^{-001}	0.1218 e^{-003}	0.1195 e^{-007}	0.1938 e^{-0010}
110	0.4802 e^{-001}	0.1116 e^{-003}	0.1199 e^{-007}	0.1878 e^{-0010}
115	0.2867 e^{-001}	0.0771 e^{-003}	0.0951 e^{-007}	0.5546 e^{-0010}
120	0.0000 e^{-001}	0.0000 e^{-003}	0.0006 e^{-007}	0.5821 e^{-0010}

4. CONCLUDING REMARKS

The numerical solutions obtained shows that this method is consistent and numerically stable since the error tend towards zero as we increase number of degree n of approximation. This method has algorithm which make it easier since the computation are done faster and accurately on the computer system for large n. More so, this proposed methods has been tested on various examples which revealed that this method is very effective.

REFERENCES

[1] B. Mercier *An Introduction to the Numerical Analysis of Spectral Methods*,Springer-Verlag, 1989.

[2] J.P Boyd, *Chebyshev and Fourier spectral method.* , Dover, New York, 2001

[3] C. Canuto, M. Y. Hussaini, A. Quarteroni, Th.A. Zang, *Spectral methods fundamentals in single domains*, 2006.

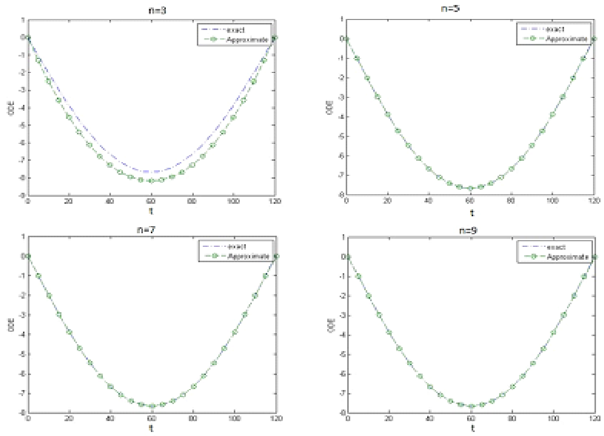

Fig. 3. Bifurcation plot for a variable viscosity when $\Gamma = \Lambda = 1$.

[4] S. N. Bernstein, *Demonstration du theoreme de weierstrass fondee sur lecalcul des probabilities*, Comm. Soc. Math. Kharkov **13** (1) 1-2, 1912.
[5] D. Gottlieb, S. Orszag, *Numerical analysis of spectral methods theory and application*, SIAM, Philadelphia **26**, 2005.
[6] L.N. Trefethen, *Spectral methods in matlab*, Society for Industrial and Applied Mathematics, Philadelphia **160** , 2000.
[7] M. Miletic, A. Arnold, *An euler-bernoulli beam equation with boundary control: Stability and dissipative fem*, ASC Report **12** , 2013.
[8] S. R. Gunakala, D. M. G. Comissiong, K. Jordan, A. Sankar, *A finite element solution of the beam equation via matlab*, International Journal of Applied Science and Technology **2** (8) 80, 2012.
[9] Vidar Stiernstrom, *Numerical simulation of the dynamic beam equation using the sbp-sat method*, Uppsala Universitat **60** (1) 70-81, 2014.
[10] K.S. Thankane and T. Sty, *Finite difference method for beam equation with free ends using mathematica*, Southern Africa Journal of Pure and Applied Mathematics **60** (1) 70-81, 2009.
[11] Siang - Yu Tsai, *Numerical computation for nonlinear beam problems*, Master's thesis, National Sun Yat - sen University, Kaohsiung, Taiwan, 2009.

DEPARTMENT OF MATHEMATICS, FEDERAL UNIVERSITY LAFIA, LAFIA , NIGERIA
E-mail address: `olagunjuas@gmail.com`

DEPARTMENT OF MATHEMATICAL SCIENCES, FEDERAL UNIVERSITY LAFIA, LAFIA, NIGERIA
E-mail addresses: `mosaiclaw22@gmail.com`