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NUMERICAL SOLUTION OF BAGLEY-TORVIK FRACTIONAL DIFFERENTIAL EQUATION

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Abstract

This article proffers solution to Bagley-Torvik fractional differential equation via the use of linear combination of unknown coefficients α_i and shifted Chebyshev polynomial $T*_i(x)$ as basis function. The usability of the proposed technique is demonstrated on set of examples and result obtained shows the effectiveness of the technique

Keywords: Bagley-Torvik equation; Chebyshev polynomials; Fractional differential equation

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1. Introduction

Fractional differential equation is a non-integer order differential equation of the form:

$$\left(A_0 D^{\beta_n} + A_1 D^{\beta_{n-1}} + . . . + A_n D^{\beta_1}\right) y(t) = f(t) \qquad (1)$$

Where $\beta_n > \beta_{n-1} > ... > \beta_1, \quad A_0 \neq 0, \ \beta_n \geq 1, \ and \ \beta_k \in R^+, A_k \in R, k = 1,...,n$

The function $f(t)$ belongs to the space $L^2(\Omega)$ and $\Omega = [0, T]$, $T \in R^+$.

This non integer order differential equation has enjoyed a wide spectrum of applications in Economics, Biology, Physics, Chemistry, and Engineering fields. For example, it has successfully been used to provide reliable and effective model for diffusion process, financial market behaviour, blood flow phenomena, viscoelasticity and a host of other problems [1-3].

For the purpose of these diverse applications, effective solution techniques are always needed for solution to these models.

Bagley-Torvik equation (BTE) is a fractional differential equation that finds application in the model of viscoelasticity problems. Typically, the BTE considered in this work takes the form:

$$A_0 y''(t) + A_1 D^{3/2} y(t) + A_2 y(t) = f(t) \qquad (2)$$

Subject to initial-boundary conditions:

$$\begin{aligned} \gamma_0 y(0) + \gamma_1 y'(0) &= \rho_0 \\ \gamma_3 y(T) + \gamma_4 y'(T) &= \rho_1 \end{aligned} \qquad (3)$$

where $A_0, A_1, A_2, \rho_0, \rho_1, \gamma_0, \gamma_1, \gamma_3, \gamma_4, \rho_0$ *and* ρ_1 are constants with $A_0 \neq 0$ and $f(t)$ are functions defined on the interval $a \leq t \leq b$. [2, 3]

The equation has been solved both analytically and numerically by the use of different methods such as: Adomian decomposition [4, 5], Taylor collocation [6], Adams predictor-corrector approach [7], differential transform method. Podlubny [8] prescribed discretization of the fractional derivatives using matrix approach. On a relatively new note, Fakhrodin [1] applied Chebyshev wavelet operational matrix while on the other hand Saha [4] introduced Haar wavelet operational matrix of general order for the solution of BTE.

In this work, we proffer a less cumbersome but effective approach to obtain numerical solution to inhomogeneous BTE in (2). The approach demands that BTE be converted to system of algebraic equations whose numerical values when substituted into the assumed solution gives a very simple solution to BTE.

2. Caputo Fractional Derivative

In the solution of fractional differential equations, there is always a need to resolve the non-integer derivative contained in the equation, this goes with a lot of standard definitions for the fractional derivatives such as Grunwald-Letnikov derivatives, Riemann-Liouville Fractional derivative, Caputo Fractional derivative and a host of others [8 – 15]. The effectiveness of these definitions has been established; however few limitations exist for some of them, for instance, Riemann-Liouville's definition leads to initial conditions containing the limit values of the Riemann-Liouville fractional derivatives at the lower terminal $t=a$. As observed by Podlubny [8], in spite of the fact that initial value problems with such initial conditions can be successfully solved mathematically, their solutions are practically useless because there is no known physical interpretation for such type of initial conditions. To resolve this limitation, Caputo [8] proposed, a definition for fractional derivative as:

$$
{}_aD_t^{\alpha} f(t) = {}_CD_{a,t}^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t \dfrac{f^{(n)}(\tau)\, d\tau}{(t-\tau)^{\alpha+1-n}}, & n-1<\alpha<n\in N \\ \dfrac{d^n}{dt^n} f(t), & \alpha = n \in N \end{cases} \tag{4}
$$

It is noted that for $\alpha \to n$ the Caputo derivative resulted into integer order n^{th} derivative of the function $f(t)$ i. e.

$$
\lim_{\alpha\to n} {}^CD^{\alpha} f(t) = \lim_{\alpha\to n}\left(\frac{f^{(n)}(a)(t-a)^{n-\alpha}}{\Gamma(n-\alpha+1)} + \frac{1}{\Gamma(n-\alpha+1)} \int_a^t f(t-\tau)^{n-\alpha} f^{(n+1)}(\tau)\, d\tau \right)
$$

$$
= f^n(a) + \int_a^t f^{(n+1)}(\tau)\, d\tau = f^{(n)}(t), \qquad n = 1, 2, 3, \ldots \tag{5}
$$

The fractional part of equation (2) is thus resolved in Caputo sense in the form:

$$
D^{3/2} y(t) = J^{\frac{1}{2}} y^{(2)}(t) \tag{6}
$$

Where $$J^{\frac{1}{2}} g(t) = \frac{1}{\Gamma(\frac{1}{2})} \int_0^t (t-u)^{-1/2} g(u)\, du$$

3. Shifted Chebyshev Polynomials

BTE exists in interval [0, T] rather than the natural range [-1, 1], with the usual transformation as discussed in [12, 16], Chebyshev polynomial shifted into this interval gives:

$$T_n^*(x) = \cos\left\{n\cos^{-1}\left(\frac{2x-T}{T}\right)\right\}$$

The 3 term recursive relation gives:

$$T_{n+1}^*(x) = 2\left[\frac{2x-T}{T}\right]T_n^*(x) - T_{n-1}^*(x)$$

From this, we have

$$T_0^*(x) = 1$$

$$T_1^*(x) = \frac{2x}{T} - 1$$

$$T_2^*(x) = \frac{8x^2}{T^2} - \frac{8x}{T} + 1$$

$$T_2^*(x) = \frac{32x^3}{T^3} - \frac{48x^2}{T^2} + \frac{18x}{T} - 1$$

The orthogonality condition and the analytic form of $T_n^*(x)$ are clearly given in [1,12,16]

4. Numerical Techniques

The technique employed involves writing the solution as a linear combination of shifted Chebyshev polynomial and unknown coefficients α_i to give

$$\bar{y} = \sum_{i=0}^{N}\alpha_i T_i*(t) \tag{7}$$

Where $T_i*(x)$ represents Chebyshev polynomial of the first kind shifted into interval [0 T]. The coefficient α_i are unknown adjustable coefficients/parameters frequently called generalized coordinates [12]

The approach involves substituting (7) into (2) which becomes:

$$R(t) \Rightarrow A\frac{d^2}{dt^2}\left(\sum_{i=0}^{n}a_iT_i^*(t)\right) + B\left(\frac{1}{\Gamma(1/2)}\int_0^t (t-s)^{\frac{-1}{2}}y_n''(s)ds\right) + C\left(\sum_{i=0}^{n}a_iT_i^*(t)\right) = f(t) \tag{8}$$

It is to be noted that the fractional derivative is expressed via Caputo's definition as illustrated in equation (4).

It is also to be noted that if $f(x)$ is not a polynomial, it is converted using a technique discussed in [16].

The coefficients of t on both sides of (8) are thereafter equated to each other. This produces a number of equations in t. In addition to these equations, two other equations are arrived at by imposing the initial-boundary conditions on (7) such that:

$$\gamma_0 \sum_{i=0}^{N} \alpha_i T_i^*(0) + \gamma_1 \sum_{i=0}^{N} \alpha_i\, T*'(0) = \rho_0 \tag{9}$$

$$\gamma_3 \sum_{i=0}^{N} \alpha_i T_i^*(T) + \gamma_4 \sum_{i=0}^{N} \alpha_i\, T*'(T) = \rho_1 \tag{10}$$

Equations arrived at in (8) together with equations (9) and (10) form a system of n + 1 equations. These are solved with the use of any algebraic system solver and the resulting numerical values of $\alpha_i's$ are substituted into (7) and that gives the needed approximate solution.

5. Examples

Some examples of BTE are considered to illustrate the simplicity, efficiency and accuracy of the proposed method.

Example 5.1

Consider the following boundary value problem in the case of the inhomogeneous Bagley-Torvik equation [1].

$$D^2\, y(t) + D^{\frac{3}{2}}\, y(t) + y(t) = t^2 + 4\sqrt{\tfrac{t}{\pi}} + 2$$
$$y(0) = 0$$
$$y(5) = 25$$

Where the exact solution of this problem is $y\,(t) = t^2$. For the solution to this problem, we apply the method described in this work with $n = 2$. A system of 4 linear equations is arrived at, out of which 1 is selected (With the criteria of being the equation that retains the highest number of unknowns). This is conjunction with the equations from the boundary conditions are solved and gives:

$\alpha_0 = 18.75$
$\alpha_1 = 12.5$
$\alpha_2 = 3.125$

Substituting these into (7) gives the numerical solution to problem 5.1. When compared to the exact, this solution yields zero discrepancies across the interval of consideration.

Example 5.2

Consider the BTE

$$D^2 y(t) + D^{3/2} y(t) + y(t) = 1 + t$$

The boundary condition is given as $y(0) = y'(0) = 1$

The exact solution is $y(t)\, 1 + t$. With $n = 2,$ the solution of this problem gives:

$\alpha_0 = 1.5,\ \alpha_1 = 0.5\ \ and\ \ \alpha_2 = 0.$ Putting these into (7) equals to the exact solution

Example 5.3

Solve the BTE

$D^2 y(t) + D^{3/2} y(t) + y(t) = 1 + t, \quad 0 \le t \le 1$

with boundary conditions $y(0) = 1$ *and* $y(1) = 2$.

The analytic solution is $y(t) = 1 + t$

When applied, the discussed method yields:

$\alpha_0 = 3,\ \alpha_1 = 0.5\ \ and\ \ \alpha_2 = 0$ when $n = 2,$

putting these into (7) also gives the exact solution.

Example 5.4

Consider the fractional boundary value problem [17]:

$$y''(x) + 0.5\, D^{\beta} y(x) + y(x) = 3 + x^2\left(\frac{x^{-\beta}}{\Gamma(3-\beta)} + 1\right)$$

with boundary condition $y(0) = 1,\ y(1) = 2$

The exact solution of this problem is $y = x^2 + 1$. We solve this problem for $\beta = 3/2$ and arrived at the solution with zero error when $n = 2$, the numerical coefficients are

$\alpha_0 = 2.75,\ \alpha_1 = 0.5\ \ and\ \ \alpha_2 = 0.125$ which are substituted into (7) to yield the approximate solution.

Conclusion

From numerical results obtained in examples 5.1 – 5.4, it is obvious that the proposed technique gives exact solution with a very minimal computational effort ($n = 2$). It is equally noted that $\alpha_i \to 0$ as $n \to \infty$ such that at the particular n where the exact solution is attained, we observe $\alpha_i = 0$ for higher order n.

In conclusion, the proposed technique as applied in finding numerical solution to Bagley-Torvik equation is simple, easy to automate and highly efficient with minimal computational time.

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