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Numerical Solution of Nonlinear Singular Ordinary Differential Equations Arising in Biology via Operational Matrix of Bernstein Polynomials

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Abstract

In this paper, a numerical method based on Bernstein polynomial defined on the interval [0 1] is applied for the solution of nonlinear singular differential equation that arises in modelling of some diseases and biology. These nonlinear problems are solved by first writing the solution as a linear combination of Bernstein basis functions. Established properties of Bernstein polynomials together with operation matrix of its derivatives are employed to express each term in the given equation in easy-to-handle form, such that the problem of finding solution of nonlinear differential equations is reduced to problems of finding solution to algebraic equations which are easily handled by an algebraic solver- MATLAB. The method is computationally efficient and applications are demonstrated via some numerical examples.

Keywords: Keywords: Nonlinear Singular Boundary Value Problem; Bernstein polynomials; Physiology; Biology.

1. Introduction

Mathematical models of most real life problems cannot be solved via analytical method of solution as closed form solution does not exist for several of this class of problem, instead they must be approached and solved through the use of computational methods that result into approximate solution. The use of polynomials in approximation is diverse and of great importance, the availability of fast computing machines and numerous application software has further deepened the use of these numerical tools in virtually all aspect of numerical approximation. The fact that these can be easily differentiated, integrated and pieced together to form spline curves makes it possible to achieve any level of desired accuracy in function approximation with polynomials.

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A good number of authors have studied Bernstein polynomials and their properties, the traits exhibited by this polynomial and results obtained from their works justify the usefulness of these polynomials in the field of function approximation. Properties such as positivity, continuity, recursion's relation, symmetry and unity partition of the basis set over the interval [0, 1] have also been studied (see [1], [2], [3]),

The aim of this paper is to deploy these polynomials in the application of Collocation method in finding numerical solution to a class of nonlinear singular boundary value problems (SBVPs) which arises in biology and physiology problems. The general form of this problem is given as:

$$y''(x) + \left(a + \frac{m}{x}\right) y' = f(x,y) \qquad 0 \le x \le 1 \tag{1}$$

$$\alpha_1 y(0) + \beta_1 y'(0) = \gamma_1 \tag{2a}$$

$$\alpha_2 y(1) + \beta_2 y'(1) = \gamma_2 \tag{2b}$$

For $(x, y) \in \{(0,1) \times \Re\}$, $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous and $\frac{\partial f}{\partial y} \ge 0 \;\; \forall\, x \;\; 0 \le x \le 1.$ $\alpha, \beta \ge 0$ and γ are finite real constants.

SBVP of this nature arises as governing equation to several problems in biology and physiology. Cases where $m = 0,1,2$ and $a = 0$ arise in the study of various problems in tumour growth with linear or nonlinear $f(x,y)$. This equally arises in the study of steady-state oxygen diffusion in a spherical cell with Michaelis-Menten uptake kinetics (See [4,5,6]) where the equation takes the form:

$$(x^{\sigma} y'(x))' = x^{\sigma} f(x, y(x)), \qquad \sigma \ge 0 \tag{3}$$

with $\sigma = 2$ and $f(x,y) = \dfrac{N y}{y + \mu}$ where N *and* μ are positive real constants.

In heat conduction model in human head. $\sigma = 2$, $f(x,y) = -a_0 e^{-b_0 y}$ where b_0 and a_0 are positive real numbers ([5], [6])

A good number of researchers have studied this problem, each with different solution approaches. The technique in this work involves the use of Bernstein polynomials in the application of the age-long numerical method called collocation. This method since its introduction has undergone various level of improvements resulting from the use of different types of polynomial, techniques and collocation points [7],

[8]. The method herein discussed involves writing the solution form as a linear combination of Bernstein polynomials with coefficients of approximation. With the aid of operational matrix of derivatives, the entire nonlinear differential equation is written in algebraic equations.

1. Bernstein Polynomials

Named after a Russian mathematician (Sergei Bernstein), Bernstein polynomials is a linear combination of Bernstein basis polynomials. The form used in this work which is valid for interval $0 \leq x \leq 1$ is written as:

$$B_n(x) = \sum_{i=0}^{n} c_i \, b_{i,n}(x) \tag{4}$$

Where $b_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad i = 0, 1, \ldots, n$ (5)

The binomial coefficient is expressed as $\binom{n}{i} = \dfrac{n!}{i!(n-i)!}$.

There are $n + 1$ nth-degree Bernstein polynomials. For mathematical convenience, we usually set $b_{i,n}(x) = 0 \;\; if \;\; i < 0 \;\; or \;\; i > n$.

For function *f*, a real-valued function defined and bounded on the interval [0 , 1], the Bernstein polynomial of *f* is defined by:

$$B_n(f) = B_n(x; f) = \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right) b_{i,n}(x) \tag{6}$$

2. Some Properties of Bernstein Polynomials

For the sake of easy referencing, some properties of Bernstein polynomials and derivations that are of direct relevance to this work are given in this section:

2.1 Recursion

This refers to a blend of two Bernstein polynomials of degree *n* -1 to yield Bernstein polynomial of degree *n*. That is , the *k*th nth-degree Bernstein polynomial can be written as:

$$b_{k,n}(x) = (1-x)b_{k,n-1}(x) + x b_{k-1,n-1}(x) \qquad (7)$$

$$b_{0,0}(x) = 1$$

2.2 Positivity

Bernstein polynomials of degree n, each is non-negative on $0 \le x \le 1$ and are strictly positive on the open interval $0 < x < 1$

2.3 Partition of Unity

The k + 1 Bernstein basis polynomials for a Bernstein polynomial of degree k form a partition of unity. That is:

$$B_k(x) = \sum_{i=0}^{n} b_{i,n}(x) = b_{0,n} + b_{1,n} + \ . \ . \ . \ + b_{n,n} = 1, \qquad 0 \le x \le 1$$

2.4 Symmetry $b_{i,n}(x) = b_{i,n-i,n}(1-x)$

2.5 Degree elevation

Any of the lower-degree Bernstein polynomials (degree < n) can be expressed as a linear combination of Bernstein polynomials of degree n. That is,

$$b_{i,n-1}(x) = \left(\frac{n-i}{n}\right) b_{i,n}(x) + \left(\frac{i+1}{n}\right) b_{i+1,n}(x)$$

which expresses a Bernstein polynomial of degree n - 1 in terms of a linear combination of Bernstein polynomials of degree n.

2.6 Power basis element in Bernstein polynomials form

Each power basis element are in this work written as linear combination of Bernstein polynomials via degree elevation which yields:

$$x^k = \sum_{i=k-1}^{n-1} \frac{\binom{i}{k}}{\binom{n}{k}} b_{i,n}(x)$$

2.7 Bernstein polynomials as a basis

The Bernstein polynomials span the space of polynomials and they are linearly independent, Therefore, the Bernstein polynomials of order n form a basis for the space of polynomials of degree less than or equal to n.

The proof for all of the above assertions can be found in [1].

3. Matrix representation for Bernstein polynomials

From equation (4), we have $B_n(x) = c_0 b_{0,n}(x) + c_1 b_{1,n}(x) + \ . \ . \ . \ + c_n b_{n,n}(x)$

Writing this as a dot product of two vectors, we have:

$$B_n(x) = \begin{bmatrix} b_{0,n}(x) & b_{1,n}(x) & . \ . \ . & b_{n,n}(x) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} \qquad (7)$$

$$B_n(x) = \begin{bmatrix} 1 & x & x^2 & \cdots & x^n \end{bmatrix} \begin{bmatrix} b_{0,0} & 0 & 0 & \cdots & 0 \\ b_{1,0} & b_{1,1} & 0 & \cdots & 0 \\ b_{1,0} & b_{2,1} & b_{2,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n,0} & b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \qquad (8)$$

Where $b_{i,j}$ are the coefficients of the power basis that are used to determine the respective Bernstein polynomials. The matrix is lower triangular.

3.1 Relationship between Bernstein basis polynomials matrix and their derivatives

The derivatives of nth-degree generalized Bernstein basis polynomials are given by:

$$\frac{d}{dx} b_{i,n}(x) = \frac{n}{b-a}\left[b_{\,i-1,n-1}(x) - b_{i,n-1}(x) \right], \ for \ \ i = 0,1, \ \cdots n. \qquad (9)$$

The relation between generalized Bernstein basis polynomials matrix and their derivatives given in the above equation (9) is of the form:

$$\mathbf{B}^{(k)}(x) = \mathbf{B(x)}\,\mathrm{N}^{k} \ ; \ k = 1,\ 2,\ \cdots,\ m$$

The elements of $(n+1) \times (n+1)$ matrix **N =** (m_{ij}), *i, j*= 0, 1, . . ., N are defined by:

$$m_{ij} = \frac{1}{b-a}\begin{cases} n-i, & if \quad j=i+1 \\ 2i-n, & if \quad j=i \\ -i, & if \quad j=i-1 \\ 0, & otherwise \end{cases}$$

Where

$$\mathbf{B}(x) = \lfloor b_{0,n}(x) \quad b_{1,n}(x) \quad \cdots \quad b_{n,n}(x) \rfloor$$

$$\mathbf{B}'(x) = \lfloor b'_{0,n}(x) \quad b'_{1,n}(x) \quad \cdots \quad b'_{n,n}(x) \rfloor$$ and

$$\boldsymbol{N} = \frac{1}{b-a}\begin{bmatrix} -n & n & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2-n & n-1 & \cdots & 0 & 0 & 0 \\ 0 & -2 & 4-n & \cdots & 0 & 0 & 0 \\ 0 & 0 & -3 & \cdots & 0 & 0 & 0 \\ \vdots & & & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & n-4 & 2 & 0 \\ 0 & 0 & 0 & \cdots & 1-n & n-2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -n & n \end{bmatrix} \qquad (10)$$

4. Method of Solution

The basic technique adopted in this method is to seek a solution for (1) in the form of Bernstein polynomials equation (4). The singular differential equation (1) is satisfied by the Bernstein polynomials at the collocation points $x_i \in [0\ ,\ 1];\quad i = 1, 2, \cdots n-1$. For the purpose of this work, x_i is expressed as:

$$x_i = \frac{i}{n}, \quad i = 1, 2, \cdots n-1 \quad . \qquad (11)$$

The first approach involves substituting the solution form:

$$y_n(x, c) = B_n(x) = \sum_{i=0}^{n} c_i b_{i,n}(x)$$

into (1) to yield

$$\frac{d}{dx^2}\left(\sum_{i=0}^{n} c_i b_{i,n}(x)\right) + \left(a + \frac{m}{x}\right)\frac{d}{dx}\left(\sum_{i=0}^{n} c_i b_{i,n}(x)\right) = f(x, y) \qquad (12)$$

With the use of properties discussed in section 3 in conjunction with expression for derivatives and matrix representation, collocating equation (10) at points (11) yields matrix equation of the form:

$$\sum_{k=0}^{m} \mathbf{C}_k \, \mathbf{B} \mathbf{N}^k \, \mathbf{Y} = \mathbf{F} \tag{13}$$

The matrices are**:**

$$\mathbf{Y} = [\, y\,(\tfrac{i}{n})\,]\,,\quad \mathbf{C}_k = diag\,[\,c_k\,(x_i\,)\,]\,,\ \mathbf{B} = [\,b_{\,jn}\,(\,x_i\,)\,]\ and\ \mathbf{F} = [\,(x_i\,,\,y_i\,)\,]\,;\, i\,, j = 0,1,\cdots, n. \tag{14}$$

Equation (13) is then solved via algebraic solver – MATLAB to give c_k which are thereafter substituted into (4) to give the solution to equation (1).

It is to be noted that whenever $f\,(x\,,y)$ in (1) takes a nonlinear term, solving (13) yields several sets of c_k depending on the nature of the nonlinearity. It is observed thereafter the optimal set of c_k are the ones with least variance.

1. Numerical Results

With the use of numerical algorithm written in MATLAB 7.9, the method herein discussed are applied on 3 problem. In cases where exact solution is unknown the results are compared with results produced by other methods, while in other cases, results are compare with analytical solution and corresponding errors outlined.

Example 6.1

Solve the following oxygen diffusion problem [5]:

$$(x^{\alpha}\,y'(x))' = x^{\alpha}\left(\frac{N_1 y(x)}{y(x)+K_1}\right), \qquad 0 < x < 1,$$

With boundary conditions:

$$y'(0) = 0, \qquad 5y(1) + y'(1) = 5$$

Where N_1 and K_1 are positive constants involving the reaction rate and Michealis constant (see [9], [10], [11]). Thus, $N_1 = 0.76129\ \ and\ \ K_1 = 0.03119$

Table 1. Approximate solutions for Example 6.1

x	Present Method With n=10	Method in [4] With m = 15	Method in [10] With n = 20
0	0.82848329	0.82848329	0.828483295
0.1	0.829706093	0.829706092	0.829706097
0.2	0.833374733	0.833374734	0.833374738
0.3	0.839489914	0.839489914	0.839489918
0.4	0.848052785	0.848052785	0.848052789
0.5	0.859064928	0.859064927	0.859064928
0.6	0.872528319	0.87252832	0.872528316
0.7	0.888445305	0.888445306	0.888445299
0.8	0.906818548	0.906818548	0.906818542
0.9	0.927650988	0.927650988	0.927650983
1	0.950945798	0.950945798	0.950945795

Example 6.2

Consider the SBVP:

$$y''(x) + \tfrac{1}{x} y'(x) = -e^{y},$$
$$y'(0)=0, \quad y(1)=0$$

With exact solution

$$y(x) = 2\ln\left(\frac{c+1}{cx^2+1}\right), \quad \text{where } c = 3-2\sqrt{2}$$

Table 2. Table of errors for Example 6.2

x	Present Method With n=10	Method in [4] With m = 20	Method in [10] With n = 20
0	3.33E-16	2.22E-16	2.00E-06
0.1	4.44E-16	3.33E-16	1.99E-06
0.2	9.99E-16	4.99E-16	1.97E-06
0.3	1.11E-16	2.77E-16	1.94E-06
0.4	4.44E-16	3.88E-16	1.83E-06
0.5	4.16E-16	4.44E-16	1.78E-06
0.6	1.11E-16	8.32E-17	1.67E-06
0.7	0	3.33E-16	1.34E-06
0.8	0	2.35E-16	9.20E-07
0.9	0	1.24E-16	4.57E-07
1	0	7.11E-18	0

Example 6.3

Consider the following singular two-point boundary value problem [5,12]

$$y''(x) + (1 + \tfrac{h}{x})\, y'(x) = \frac{5x^3(5x^5 e^y - h - 4 - x)}{4 + x^5}$$

$$y(0) = \ln(\tfrac{1}{4}), \quad y(1) + 5y'(1) = \ln(\tfrac{1}{5}) - 5,$$

with exact solution $y(x) = \ln\left(\frac{1}{4 + x^5}\right)$

Table 3. Table of maximum absolute error in solution of example 6.3

N	h = 0.25	h = 0.75	h = 1	h = 2
10	1.44 E-03	2.36E-04	2.36E-04	4.86E-05
15	8.11E-07	2.72E-07	7.63E-08	1.02E-08
20	9.26E-11	5.18E-11	4.11E-12	9.80E-13

Example 6.4

Consider the following nonlinear SBVPs:

$$\left(x^2 y'(x)\right)' = x^2 y^5(x), \quad 0 < x < 1,$$
$$y'(0) = 0, \quad y(1) = \tfrac{\sqrt{3}}{2}.$$

The anlytical solution is $y(x) = \sqrt{\frac{3}{3 + x^2}}$

Table 4. Table of absolute error in solution of example 6.4 for n=10.

X	n = 10	n = 15
0	1.96e-08	2.31e-13
0.1	1.93e-08	1.73e-13
0.2	1.85e-08	1.51 e-13
0.3	1.72e-08	1.72 e-13
0.4	1.54 e-08	1.28 e-13
0.5	1.31 e-08	1.12e-13
0.6	1.03 e-08	1.08e-13
0.7	7.22 e-08	9.11e-13
0.8	3.76 e-08	6.49e-13
0.9	2.55 e-08	4.01e-13
1.0	0	0

7. Conclusion

Solution techniques based on collocation method has been discussed and applied for numerical solution of nonlinear singular boundary value problem arising in biology and physiological problems using Bernstein polynomials.

From the results outlined above, it is obvious that this method produced good results with not-too-large n, hence minimal computational cost both in resources and time.

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