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Chebyshev methods for the numerical solution of fourth-order differential equations

O. A TAIWO[1]* and A. S. OLAGUNJU[2]

[1]Department of Mathematics, University of Ilorin, Ilorin Nigeria.
[2]Department of Mathematics, Kwara state University, Malete, Nigeria.

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We consider in this paper the application of Chebyshev polynomials in solving fourth-order differential equations and trial solution constructed as Chebyshev form of Fourier cosine series is employed. Also, formula which enables both sides of the differential equations to be expressed as sum of Chebyshev polynomials is derived. As a means of finding the numerical values of the approximant, collocation and coefficients comparison techniques are applied after the entire differential equation is converted into Chebyshev form. We seek to investigate the efficiency of these methods and the nature of problem that each can handle most effectively. For polynomial variable coefficients equations, standard formula for expressing such, in term of Chebyshev series is applied.

Key words: Collocation, coefficient comparison, trial solution, Chebyshev series.

INTRODUCTION

This paper is concerned with the approximate solution of fourth-order differential equation of the form:

$$\alpha(x)y^{iv}(x) + \beta(x)y^{iii}(x) + \delta(x)y^{ii}(x) + \varphi(x)y^{i}(x) + \mu(x)y(x) = f(x) \quad (1)$$

which is valid in interval $a \leq x \leq b$, together with sufficient conditions imposed on the dependent variable at the end points $x = a \text{ and } x = b$

$\alpha(x),\ \beta(x),\ \delta(x),\ \varphi(x),\ \mu(x),\ and\ f(x)$ *are known functions*

The work is structured to compare the efficiency of the two methods namely collocation and coefficient comparison which are applied after the given differential equations are converted into Chebyshev series. For this conversion, a formula that directly converts derivatives (up to fourth-order) into terms of Chebyshev series is derived. In contrast to the formula;

$$\left.\frac{d^pT_n}{dx^p}\right|_{x=\pm 1} = (\pm)^{n+p}\prod_{k=0}^{p-1}\frac{n^2-k^2}{2k+1} ,$$

that finds derivatives of Chebyshev polynomial in terms of independent variable x (Szego, 1975), the new formula finds the derivatives in terms of Chebyshev series T_0, T_1, T_2, T_3,….Yasuaki et al. (2007) studied the use of Chebyshev approximation for delay differential equation and applied it to problems on population dynamics. Aysegul (2008) on the other hand, solved partial differential equations (PDEs) with Chebyshev approximation formulations for partial derivatives.

To further adapt the use of Chebyshev polynomials on Equation 1, trial solutions constructed as Chebyshev form of Fourier cosine series (Fox and Parker 1968; Snyder 1966; Boyd and John 2001) is herein used.

Two techniques called collocation and coefficients comparison (Eric, 2006; Schroder, 1984; Wei and Jianzhong,

*Corresponding author. E-mail: oataiwo2002@yahoo.com.

1996) are applied as a means of finding the coefficients of the approximant after the entire differential equation has been converted into Chebyshev form. The aim of this paper is therefore, to investigate the efficiency of these methods on the class of problems considered and the nature of problem that each can handle most effectively.

CHEBYSHEV POLYNOMIALS

The Chebyshev polynomials of the first kind which is denoted by $T_r(x)$ are a set of polynomials of degree n defined as the solution to the first kind Chebyshev differential equation and is defined by

$$T_r(x) = \cos\left(r\cos^{-1}\left(\frac{2x-b-a}{b-a}\right)\right); \qquad (2)$$

and satisfies the recurrence relation

$$T_{r+1}(x) = 2\left[\frac{2x-b-a}{b-a}\right]Tr(x) - T_{r-1}(x); \quad a \le x \le b \qquad (3)$$

$a \le x \le b$ is the interval within which x is valid

CHEBYSHEV REPRESENTATION FOR ORDINARY DERIVATIVES

$$y' = a_1T_1'(x) + a_2T_2'(x) + \ . \ . \ . \ + a_nT_n'(x)$$

$$y'' = a_1T_1''(x) + a_2T_2''(x) + \ . \ . \ . \ + a_nT_n''(x)$$

$$y''' = a_1T_1'''(x) + a_2T_2'''(x) + \ . \ . \ . \ + a_nT_n'''(x) \qquad (4)$$

$$y'^{v} = a_1T_1'(x) + a_2T_2'^{v}(x) + \ . \ . \ . \ + a_nT_n'^{v}(x)$$

Derived formula for the expression of these derivatives as series of Chebyshev terms without derivatives is as follows:

$$\frac{d^n}{dx^n}T_r(x) = 2^n r \sum_{j=1}^{(r-n+2)/2}\left[\rho\prod_{k=j}^{k+n-2}(r-k)\,\phi T_{r-n-2j+2}(x)\right] \qquad (5)$$

where

$$\rho = \begin{cases} 1 & for\ n=1 \\ j & for\ n=2 \\ \dfrac{j(j+1)}{2} & for\ n=3 \\ \dfrac{j(j+1)(j+2)}{6} & for\ n=4 \end{cases}$$

and coefficient of $T_r(x)$ that is,

$$\phi = \begin{cases} \dfrac{1}{2} & for\quad r-n-2j+2=0 \\ 1, & for\quad r-n-2j+2>0 \end{cases}$$

Using the above formulae, each term of derivatives in Equation 4 are directly converted to terms of Chebyshev polynomials as follows:

Interval $-1 \le x \le 1$

For even r

$$T_r'(x) = 2r\{T_{r-1} + T_{r-3} + \ . \ . \ . \ + T_1\}$$

$$T_r''(x) = 4r\left\{(r-1)T_{r-2} + 2(r-2)T_{r-4} + 3(r-3)T_{r-6} + \ . \ . \ . \ + ...\frac{T_0}{2}\right\} \qquad (6)$$

$$T_r'''(x) = 8r\{(r-1)(r-2)T_{r-3} + 3(r-2)(r-3)T_{r-5} + 6(r-3)(r-4)T_{r-7} + \ . \ . \ . \ + ...T_1\}$$

$$T'^{v}{}_r(x) = 16r\left\{(r-1)(r-2)(r-3)T_{r-4} + 4(r-2)(r-3)(r-4)T_{r-6} + 10(r-3)(r-4)(r-5)T_{r-8} + \ . \ . \ . \ + ...\frac{T_0}{2}\right\}$$

For odd r

$$T_r'(x) = 2r\left\{T_{r-1} + T_{r-3} + \ . \ . \ . \ \frac{T_0}{2}\right\}$$

$$T_r''(x) = 4r\{(r-1)T_{r-2} + 2(r-2)T_{r-4} + 3(r-3)T_{r-6} + \ . \ . \ . + ...T_1\} \quad (7)$$

$$T_r'''(x) = 8r\{(r-1)(r-2)T_{r-3} + 3(r-2)(r-3)T_{r-5} + 6(r-3)(r-4)T_{r-7} + \ . \ . \ . + \frac{T_0}{2}\}$$

$$T'^{v}{}_r(x) = 16r\{(r-1)(r-2)(r-3)T_{r-4} + 4(r-2)(r-3)(r-4)T_{r-6} + 10(r-3)(r-4)(r-5)T_{r-8} + \ . \ . \ . + ...T_1\}$$

Note: The formulations are only valid within the interval for which they were derived that is, $-1 \leq x \leq 1$

TRIAL SOLUTION FORMULATION

$$P_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx), \quad (8)$$

With the Fourier coefficients defined by:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx \quad (9)$$

Is the least squares approximation to f(x), with unit weight function in $-\pi \leq x \leq \pi$.

From the work of Fox and Parker (1968), it is established that the cosine series

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos kx \qquad with \qquad a_k = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx \quad (10)$$

converges faster and in Chebyshev form, it is written as:

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k\ T_k(x) \qquad with \qquad a_k = \frac{2}{\pi}\int_{-1}^{1}(1-x^2)^{-1/2} f(x)T_k(x)dx \quad (11)$$

It was also noted in their work, that Equation 11 has the same convergence properties as the Fourier series for f(x), (Fox and Parker, 1968). Based on this property, our choice of trial solution Equation 11 is written as:

$$\bar{y}(x,\ a) = \frac{1}{2}a_0 + \sum_{r=1}^{N} a_r\ T_r(x) \quad (12)$$

where x represents all the independent variables in the problem, coefficients a_i are specialize coordinates called degree of freedom (DOF) $T_i(x)$ are Chebyshev polynomials in variables x.

METHOD 1: COLLOCATION METHOD

The trial solution given in Equation 12 is in this method substituted into the governing Equation 1 to yield a non-zero equation of the form:

$$\alpha(x)\sum_{r=1}^{N} a_r T_r^{Iv}(x) + \beta(x)\sum_{r=1}^{N} a_r T_r^{III}(x) + \delta(x)\sum_{r=1}^{N} a_r T_r^{II}(x) + \varphi(x)\sum_{r=1}^{N} a_r T_r^{I}(x) + \mu\left[\frac{1}{2}a_0 + \sum_{r=1}^{N} a_r T_r(x)\right] - f(x) \neq 0 \quad (13)$$

If the exact solution is substituted into Equation 1, then the whole equation will be equal to zero. But if any other function such as the formulated trial solution of Equation 12 is substituted, then the result would be a non-zero function, that is Equation 13 which is called residual equation (David, 1987).

Equation 13 is then collocated at points

$x_i = \cos\left(\frac{2_i + 1}{r+1}.\frac{\pi}{2}\right)$, $i = 0, \ . \ . \ . , \ r,$ which are the zeroes of relevant polynomial $T_{r+1}(x)$ (Fox and Parker, 1968). This yields (r+1) collocation equations.

It is worthy to note that the number of boundary conditions in addition to the number of the collocation equations must be equal to the number of the unknown a_i. By this, we avoid having over-determined and under-determined systems.

The set of equations generated from collocating Equation 13 in conjunction with equations derived from imposition of conditions on trial solution of Equation 12 are then solved, in order to get the values of a_i .These values are thereafter substituted back into the trial solution, thus, yielding the approximate solution for the problem.

METHOD 2: COEFFICIENT COMPARISON

We hereby adopt another technique of finding the coefficients a_i. Taking the non-finite form of Equation 12, we have;

$$\bar{y}_\infty(x;a) = \frac{1}{2}a_0 + \sum_{r=1}^{\infty} a_r\ T_r(x) \quad (14)$$

As Equation 14 is substituted into the governing Equation 1, we have;

$$\alpha(x)\sum_{r=1}^{\infty}a_rT_r^{Iv}(x)+\beta(x)\sum_{r=1}^{\infty}a_rT_r^{III}(x)+\delta(x)\sum_{r=1}^{\infty}a_rT_r^{II}(x)+\varphi(x)\sum_{r=1}^{\infty}a_rT_r^{I}(x)+\mu\left[\frac{1}{2}a_0+\sum_{r=1}^{\infty}a_rT_r(x)\right]=f(x) \quad (15)$$

By the use of Equation 5, the entire terms on LHS of Equation 15 are converted to terms of Chebyshev polynomials without derivatives. Also, making use of the techniques discussed in Equation 9, f(x) is also converted to terms of Chebyshev polynomials. We thereafter, factorize each of the Chebyshev terms T_r on both sides of the resulting equation and corresponding coefficients on both sides are equated.

This in conjunction with the resulting equations from imposition of boundary conditions on Equation 14 form an infinite set of equations (because no particular N is chosen yet). Since solving an infinite set of equations is not feasible, an approximation comes from solving a finite segment of these equations. The resulting values of coefficients a_i are then substituted into the trial solution. This gives the approximate solution to the problem for the particular N.

VARIABLE COEFFICIENTS EQUATIONS

For polynomial variable coefficient equations, we employ the use of the following formulae for the expression of product of Chebyshev terms and its variable coefficients in terms of Chebyshev series alone (without product).

From the recurrent relation of Chebyshev polynomials:

$$T_{r+1}=2xT_r-T_{r-1} \quad (16)$$

we have

$$xT_r=\frac{1}{2}\left(T_{r+1}+T_{r-1}\right)$$

Proceeding on this, we have;

$$x^2T_r=\frac{1}{4}\left(T_{r+1}+2T_r+T_{r-2}\right)$$

$$x^3T_r=\frac{1}{8}\left(T_{r+3}+3T_{r+1}+3T_{r-1}+T_{r-3}\right)$$

$$x^4T_r=\frac{1}{16}\left(T_{r+4}+4T_{r+2}+6T_r+4T_{r-2}+T_{r-4}\right)$$

From these, we have a general formula to generate this product as:

$$x^nT_r=\frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}T_{r+n-2i}(x) \quad (17)$$

EXPERIMENTATION OF METHOD AND NUMERICAL RESULTS

In this section, we demonstrate the use of the discussed methods and carry out numerical experiment using specific problems. The entire process is automated by a use of a computer programming language called MATLAB. Uniform degree of N is considered to facilitate an in dept study of the solution.

Example 1

Solve

$$y'^{v}-y'''+y=x^2$$

Subject to boundary conditions;

$$y(0)=y'(0)=y(1)=0$$
$$y(1)=2$$

Example 2

Solve the boundary value problem;

$$y'^{v}+81y=729x^2$$

Subject to boundary conditions;

$$y(0)=y'(0)=y''(1)=y'''(1)=0$$

Example 3

$$y^{iv}+x^3y'+2y=\frac{1}{9}x^2+\frac{2}{3}x+4$$
$$y(0)=y'(0)=y(3)=y'(3)=0$$

Example 4

Solve

$$y'^{v}-y'''+y=0$$

Subject to boundary conditions;

$$y(0)=y'(0)=y(1)=0$$
$$y(1)=2$$

Table 1. Error estimate for example 1.

N \ Method	Collocation method	Coefficient comparison method
4	4.2543E-03	4.4100E-03
6	2.8930E-03	2.8935E-03
8	2.8986E-03	2.8986E-03
10	2.8986E-03	2.8986E-03

Table 2. Error estimate for example 2.

N \ Method	Collocation method	Coefficient comparison method
4	5.9233E-05	5.9936E-05
6	2.0593E-06	3.4119E-06
8	1.7220E-08	1.8548E-05
10	8.2415E-09	8.2420E-09

Table 3. Error estimate for example 3.

N \ Method	Collocation method	Coefficient comparison method
4	2.3721E-03	1.9212E-03
6	1.4285E-03	7.1508E-04
8	4.6111E-04	1.7321E-04
10	2.0125E-05	1.8362E-05

Table 4. Error estimate for example 4.

N \ Method	Collocation method	Coefficient comparison method
4	1.7429E-03	1.2131E-03
6	5.2582E-04	4.2815E-04
8	4.8111E-04	2.7692E-04
10	1.9331E-04	8.7327E-05

CONCLUSION

Tables 1, 2, 3 and 4 show the numerical solutions in terms of maximum error obtained for the fourth-order differential equations solved. It is clearly observed that better results are obtained with the retention of more coefficients in both methods.

In practice, accuracy is determined by the consistency of successive approximations and the rate of decrease of the coefficients in the various series. This was clearly noticed in the two methods.

Also, it was very clear that method one yielded better results than method two for constant coefficient equations as in examples 1 and 2. But in variable coefficient problems and homogeneous problems such as examples 3 and 4 respectively, method two performed better.

Each of these methods is hereby recommended for the class of problem solved with the least maximum error as depicted in the examples.

Lastly, there is possibility of perturbing the two methods as discussed in Taiwo and Evans (1997) for the purpose of enhancing the result, at that level, any of these two methods may respond better, we thus concluded that the numerical methods used are feasible and effective for the class of problems considered.

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