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Perturbed Segment Domain Collocation Tau-method for the Numerical Solution of Second Order Boundary Value Problems

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Abstract

This paper concerns the numerical solution of second order boundary value problem using a perturbed segmented domain collocation –Tau method. The entire interval for which the problem is defined is partitioned into two segments. The Chebyshev polynomials shifted as the case may be, into a given interval are used as a basis for a collocation solution via the perturbed collocation method for each segment. For a given problem two different solutions are obtained, which are valid for different intervals within the domain. Numerical examples are given to illustrate the efficiency, accuracy and computational cost of the method.

Keywords: Collocation, segmented domain, auxiliary equation, partitioning, residual equations.

1.0 Introduction

This research paper has to do with the numerical solution of second order boundary problem of the form.

$$\frac{d^2u(x)}{dx^2} + \Gamma(x)\frac{du(x)}{dx} + \beta(x)\,u(x) = f(x) \qquad (1.1)$$

which is valid in some interval $a \le x \le b$ together with sufficient conditions imposed on the dependent variable at the two end points $x = a \quad and \quad x = b$.

Where x is the independent variable, $u(x)$ is an unknown function. $a(x)$, $\Gamma(x)$, $\beta(x)$ *and* $f(x)$ are known functions. The physical applications of boundary value

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problems are found in (Ref [2]). An active research has extensively been carried out on this area, with a number of numerical methods of solution developed. For instance, [10] analysed the problem using the method of weighted residuals, [9] applied Tau method using some basis functions. As a way of enhancing the result, [2] demonstrated the use of Finite Element Method (FEM) on it, with the prominent feature of partitioning the domain into any number of elements.

In this research paper have investigated the same problem using newly developed methods called segmented domain collocation Tau method (SDCM) and perturbed segmented domain collocation Tau method (perturbed SDCM). According to [11], the smaller the elements in FEM, the better the accuracy. This basic fact is what is harnessed by these methods by dividing the entire domain of the problem into two.

By this work our aim is to compare the numerical result obtained using the method mentioned above with the exact solution of some problem in the class. This is carried out alongside the numerical solution of existing methods.

2.0 Conversion techniques of Chebyshev polynomial

The Chebyshev polynomial of degree valid in interval $-1 < x \leq 1$ is defined by**:**

$$T_n(x) = \cos\{n \cos^{-1} x\} \tag{2.1}$$

The recurrence relation is given by;

$$T_{n-1}(x) = 2xT_n(x) - T_{n-1}(x) \; ; \qquad n \geq 1 \tag{2.2}$$

For the sake of problems that exist within interval other than [-1, 1]. A good $L_\infty - type$ approximation to a function $f(x)$ over $[a, b]$ is applied. For the transformation of a variable which maps $a < X < b$ into $-1 \leq x \leq 1$.

$$\text{Let} \qquad X = a x + \beta \tag{2.3}$$

where α *and* β are to be determined.

$x \in [-1, 1]$ *and* $X \in [a, b]$, then $a = -\alpha + \beta$ *and* $b = \alpha + \beta$

It follows that; $\alpha = \dfrac{b-a}{2}$ *and* $\beta = \dfrac{b+a}{2}$

Substituting this into (2.3), gives;

$$X = \frac{(b-a)x}{2} + \frac{(a+b)}{2} \Rightarrow 2X = (b-a)x + a + b$$

or

$$x \frac{(a+b-2X)}{a-b} \qquad (2.4)$$

Thus substituting equation (2.4) into equation (2.1) and (2.2) we get the general formulae for converting to any interval, where *a* and b are the bounds of the interval within which the new problem may fall.

3.0 Numerical Solution Techniques

Method 1

Segmented Domain Collocation Method (SDCM)

This method is developed as an application of standard collocation method based on principle of division of domain from Finite Element Method (FEM). In this method, the interval *xvddu*5677 ydhhdkkkd

the interval of $a \le x \le b$ solution $a \le x \le b$ $a \le x \le b$

In this method the interval $a \le x \le b$ for which the problem is defined and divided into two segments the point x_c called the point of partition , over each of these segments, a trial solution $\bar{u}^{(1)}(x:a)$ $a\bar{u}^{(2)}(x:a)$ are formulated for segment one and two respectively . The following are the step by step approach towards the solution in this method.

Step 1

Partitioning of the domain into two segments

Suppose that the interval for which the given problem is $a \le x \le b.$ This is divided into two equal portion, i.e.

Figure 1.1

Note that;

$$u(x_1) = a \, and \, u(x_2) = b \tag{3.1}$$

Are the given boundary conditions located between the two segments is the point x_c called the point o partition. By Applying $x_c \, to \, u^{(1)}(x) and \, u^{(2)}(x)$ we generate inter-segment condition respectively for segment one and two to be:

$$u^{(1)}(x_c) = c_1 \, and \, u^{(2)}(x_c) = c_2 \tag{3.2}$$

Step 2

Trial Solution Derivation

Let the trial solutions for both segment 1 and 2 be respectively denoted as:

$$\bar{u}^{(1)}(x;a) = a_0 \phi_0{}^{(1)}(x) + a_1{}^{(1)}(x) + ... + a_N \phi_N{}^{(1)}{}_{(x)}$$
$$\bar{u}^{(2)}(x;a) = a_0 \phi_0{}^{(2)}(x) + a_1{}^{(2)}(x) + ... + a_N \phi_N{}^{(2)}{}_{(x)}$$

Where x represent all the independent variables in the problem and functions $\phi_0(x), \phi,(x), .. \phi_N(x)$ are known functions called trial functions (or sometimes basis or coordinate functions). The coefficient $a_1, a_2, ... a_N$ are unknown parameter to be determined and are frequently called degrees of freedom (DOF) or sometimes generalized coordinates. See (Ref.[2]).

The construction of a trial solution consists of constructing expression for each of the trial functions in term of specific known functions.

As discus in [2] , from a practical standpoint , it is important to use function that are algebraically as simple as possible and easy to work with, because we frequently must calculate derivatives and integral of the $\phi(x)$. Powers of x are certainly the easiest for these operations, so a logical choice for trial solutions used are first few terms of a power series. I.e. polynomials of the form:

$$\bar{u}(x;a) = a_0 + a_{1\,0}x + ...+ a_N x^N \qquad (3.5)$$

Specifically, N=3 is used throughout this paper.

According to step 1, it is to be noted that for each segment , we have one Boundary Condition and one inter-segment conditions , i.e. $u(x_1) = a\, and\, u^{(1)}(x_c) = c_1\ for\ segment 1, and\, u(x_2) = b\, and\, u^{(2)}(x_c) = c_2\ for\ segment\ 2.$ For each segment , the two condition are then imposed on the power series trial solution (3.5) and the unknown parameter $a_i i = 0,1.....N$ are then reduced in number by solving for two in terms of others and substituting their values into (3.5). This is done for each segment separately.

Step 3

Residual Equation Formulation

Equation (1.1), which is the general form of second Boundary value problem ,is then written in the form:

$$\alpha(x)\frac{d^2u(x)}{dx^2} + \Gamma(x)\frac{du(x)}{dx} + \beta(x)u(x) - f(x) = 0 \qquad (3.6)$$

It is worthy of note that if equation (3.6) is valid for the entire interval $a \le x \le b,$ it is also valid for each segment of their interval.

According to [2], equation (3.6) implies that if the exact solution are substituted for u(x) on the LHS, then the RHS would be identically zero over each of the segments. But if any other function such as the approximate trial solution $u^{(1)}(x;a) and\, u^{(2)}(x;a)$ are substituted for u(x), the result would be non-zero function called the residual error for segment one and denoted by $R^{(1)}(x;a) and\, R^{(2)}(x;a)$ respectively, of which we have:

$$R^{(1)}(x;a) = \alpha(x)\frac{d^2\bar{u}^{(1)}(x;a)}{dx^2} + \Gamma(x)\frac{d\bar{u}^{(1)}(x;a)}{dx} + \beta(x)\bar{u}^{(1)}(x;a) - f(x) \neq 0 \qquad (3.7)$$

and

$$R^{(2)}(x;a) = \alpha(x)\frac{d^2\bar{u}^{(2)}(x;a)}{dx^2} + \Gamma(x)\frac{d\bar{u}^{(2)}(x;a)}{dx} + \beta(x)\bar{u}^{(2)}(x;a) - f(x) \neq 0 \qquad (3.8)$$

Where $u^{(1)}(x;a) and\, u^{(2)}(x;a)$ are defined in step 2.

Step 4

Collocating each of the residual equations

The two residual equation (3.7) and (3.8) are then collocated at point x_i called the point of collocation. This method required that for each unknown parameter a_i each of the residual equations, we choose a point x_i to be within the respective segment and not on the boundary, their location is not necessarily in any particular pattern but it might be reasonable to distribute them uniformly according to [2].

Collocation point for each segment are arrived at by the formula:

$$x_i = a + \frac{(b-a)(i)}{N}, i = 1,2,.....N-1 \qquad (3.9)$$

Where N is the degree of approximant, irrespective of the order of the differential equation being considered. At each point of the collocation the residual equation (3.7) and (3.8), which are not equal to zero, are then to be zero. i.e. on segment 1,it gives:

At $x = x_1; R^{(I)}(X_1; a) \Rightarrow 0$

At $x = x_{N-1}; R^{(1)}(x_{x-1}; a) \Rightarrow 0$

On segment 2, it gives :-

At $x = x_1; R^{(2)}(x_{x-1}; a) \Rightarrow 0$

At $x = x_{N-1}; R^{(2)}(x_{x-1}; a) \Rightarrow 0$

For a trial solution with N parameters, we therefore produce a system of N-1 linear equations for each segment.

Step 5

Auxiliary Equation and Derivation

Because of the inter-segment condition, which is equal to unknown $c_i (i = 12)$, present in the system of residual equations are c_1 and c_2 respectively for segment 1 and 2. Because of this, there is always need for one more equation to be solved for he N number of unknown in each segment. This equation is called auxiliary equation.

For segment 1, the auxiliary equation is gotten by applying the second boundary condition to segments 1 trial solution i.e $\bar{u}^{(1)}(x_c) = \bar{u}^{(2)}(x_c) = c_1 = c_2$ (3.12)

If (3.12) is satisfied, the parameters arrived at are substituted into the trial solutions for each segments that gives the approximate solution for the problem at that interval. But if $c_1 \neq c_2$ check out for the c that produces optimum solution for the problem at that point x_c for each of the segments into their respective residual equation. Then choose c which produces the residual error with smallest modulus at point x_c and use it for the two c's. the parameters a_i arrived at together with the chosen c are then substituted into the trial solution for the problem at the interval for which that segment is defined.

Method 2

Perturbed Segment Domain Tau Method (Perturbed SDCM)

The development of this method is the same with that of method 1 upto step 3 where we have equation (3.7) and (3.8) which can respectively be written as:

$$R^{(2)}(x;a) = \alpha(x)\frac{d^2\bar{u}^{(1)}(x;a)}{dx^2} + \Gamma(x)\frac{d\bar{u}^{(1)}(x;a)}{dx} + \beta\bar{u}^{(1)}(x;a) \neq f(x) \qquad (3.14)$$

And

$$R^{(2)}(x;a) = \alpha(x)\frac{d^2\bar{u}^{(2)}(x;a)}{dx^2} + \Gamma(x)\frac{d\bar{u}^{(2)}(x;a)}{dx^2} + \beta\bar{u}^{(2)}(x;a) \neq f(x) \qquad (3.15)$$

Step 4

Perturbing and collating the residual Equations

According to [2] the meaning of equation (3.14) and (3.15) is that the approximate trial solution $\bar{u}^{(1)}(x;a)\, and\, \bar{u}^{(2)}(x;a)$ or any other function other than the exact solution u(x) are substituted into the idea of the Tau-method, as conceived by Lanczos [3] is the addition to equation (3.14) and (3.15) of small perturbation term $H_u(x)$ which causes (3.14) to respectively become:

$$\alpha(x)\frac{d^2\bar{u}^{(1)}(x;a)}{dx^2} + \Gamma(x)\frac{d\bar{u}^{(1)}(x;a)}{dx} + \beta\bar{u}^{(1)}(x;a) = f(x) + H(x) \qquad (3.16)$$

$$\alpha(x)\frac{d^2\bar{u}^{(2)}(x;a)}{dx^2} + \Gamma(x)\frac{d\bar{u}^{(2)}(x;a)}{dx} + \beta\bar{u}^{(2)}(x;a) = f(x) + H(x) \qquad (3.17)$$

The perturbation term $H_n(x)$ used in this work is of the form $H_n(x) = \tau_1 T_n(x) + \tau_2 T_{n-1}(x)$,where n the order of approximant and chosen to be 4 τ_1 and τ_2 are Tau-parameters to be determined. Tn(x) is defined in equation (2.1)

For problems that are existing in domain other than interval $-1 \le x \le 1$,a conversion technique discussed in 2 is employed to arrive at the shifted Chebyshev polynomials for that interval, equation (3.16) and (3.17) are then collocated at some selected points called collocation points x_i, which are chosen to be within each segment by formula: $x_i = a + \frac{(b-a)(i)}{N+2}, i = 1, 2, \ldots N+1$ where a and b are the bounds for each segment.

The system of N+1 linear equations produced in the process of collocating each equation (20) and (21), in conjunction with the auxiliary equation gotten in the same way as that of method one, are solved simultaneously to arrive at numerical values for the unknowns a_i and c_i. These values are then substituted into our approximate trial solutions.

4.0 Numerical examples

Example 1

Solve the boundary value problem whose governing equation is $(x+1)\frac{d^2u}{dx^2} + \frac{du}{dx} = 0$ with the interval $1 \le x \le 2$ and with the boundary equation u(1) = 1and $\left[-(x+1)\frac{du}{dx}\right] = 1$

The exact solution is ; $u(x) = 1 - \ln\left[(x+1)/2\right]$

Example 2

Solve the boundary problem: $x^3\frac{d^2u}{dx^2} + x^2\frac{du}{dx} - 2 = 0,$ within the interval $1 \le x \le 2$ with boundary conditions: $u(1) = 2\ and \left[-x\frac{du}{dx}\right]_{x=2} = \frac{1}{2}$. The analytical solution is $u(x) = \frac{2}{x} + \frac{\ln x}{2}$

Example 3

Solve the boundary value problem $\frac{12x^2d^2u}{dx^2} + \frac{24xdu}{dx} = -30x^4 + 204x^3 - 351x^2 + 110x,\ 0 \le x \le 1$ with the boundary conditions u(0)=1 and u(1)=2, the exact solution is :- $\frac{1}{24}(-3x^4 + 34x^3 - 117x^2 + 110x + 24)$

Table 1: Errors for example 1

X	**Standard**	**Perturbed**	**SDCM**	**Perturbed**

	Collocation	**Collocation**		**SDCM**
1.0	0	0	0	0
1.1	4.0×10^{-4}	3.20×10^{-3}	5.0×10^{-4}	1.0×10^{-4}
1.2	7.0×10^{-4}	5.90×10^{-3}	9.0×10^{-4}	2.0×10^{-4}
1.3	1.0×10^{-3}	8.20×10^{-3}	1.3×10^{-3}	3.0×10^{-4}
1.4	1.1×10^{-3}	1.00×10^{-2}	1.5×10^{-3}	3.0×10^{-4}
1.5	1.3×10^{-3}	1.16×10^{-2}	1.7×10^{-3}	4.0×10^{-4}
1.6	1.6×10^{-3}	1.29×10^{-2}	1.9×10^{-3}	4.0×10^{-4}
1.7	1.8×10^{-3}	1.39×10^{-2}	1.7×10^{-3}	3.0×10^{-4}
1.8	2.0×10^{-3}	1.46×10^{-2}	1.5×10^{-3}	3.0×10^{-4}
1.9	2.2×10^{-3}	1.51×10^{-2}	9.0×10^{-4}	3.0×10^{-4}
	2.3×10^{-3}	1.53×10^{-2}	1.0×10^{-2}	3.0×10^{-4}

Table 2: errors for example 2

x	**Standard Collocation**	**Perturbed Collocation**	**SDCM**	**Perturbed SDCM**
1.0	0	0	0	0
1.1	1.97×10^{-2}	1.35×10^{-2}	2.64×10^{-2}	9.4×10^{-2}
1.2	3.17×10^{-2}	2.08×10^{-2}	4.36×10^{-2}	1.47×10^{-2}
1.3	4.02×10^{-2}	2.61×10^{-2}	5.47×10^{-2}	1.86×10^{-2}
1.4	4.76×10^{-2}	3.13×10^{-2}	6.03×10^{-2}	2.2×10^{-2}
1.5	5.49×10^{-2}	3.74×10^{-2}	6.03×10^{-2}	2.45×10^{-2}
1.6	6.26×10^{-2}	4.45×10^{-2}	6.11×10^{-2}	2.25×10^{-2}
1.7	7.02×10^{-2}	5.21×10^{-2}	6.15×10^{-2}	2.11×10^{-2}
1.8	7.21×10^{-2}	5.92×10^{-2}	6.20×10^{-2}	2.05×10^{-2}
1.9	8.21×10^{-2}	6.45×10^{-2}	6.24×10^{-2}	2.05×10^{-2}
	8.41×10^{-2}	6.67×10^{-2}	6.27×10^{-2}	2.06×10^{-2}

Table 3: errors for example 3

x	**Standard Collocation**	**Perturbed Collocation**	**SDCM**	**Perturbed SDCM**
0	0	0	0	0
0.1	2.5×10^{-3}	7.0×10^{-3}	3.1×10^{-3}	6.1×10^{-3}
0.2	3.2×10^{-3}	9.5×10^{-3}	4.3×10^{-3}	8.4×10^{-3}
0.3	3.2×10^{-3}	9.0×10^{-3}	4.6×10^{-3}	8.2×10^{-3}
0.4	3.0×10^{-3}	7.0×10^{-3}	4.7×10^{-3}	6.8×10^{-3}
0.5	3.2×10^{-3}	4.6×10^{-3}	4.9×10^{-3}	4.8×10^{-3}
0.6	3.7×10^{-3}	2.3×10^{-3}	3.2×10^{-3}	2.5×10^{-3}
0.7	4.3×10^{-3}	7.0×10^{-4}	1.8×10^{-3}	8.0×10^{-4}
0.8	4.5×10^{-3}	0	1.0×10^{-3}	0
0.9	3.5×10^{-3}	1.0×10^{-4}	6.0×10^{-4}	1.0×10^{-4}
1.0	0	0	0	0

5.0 Conclusion

Table 1-3 shows the numerical solutions in terms of the maximum errors obtained for the second order boundary value problems considered in this paper. It is clearly observed that of the four method used, the perburted Domain Collocation-Tau Method (Perturbed SDCM) produced

results with least errors than the other three methods. Also it is noticed that as N increases, the algebraic system of linear equations to be solved also increases. For example, When N=3 , the perturbed domain collocation method resulted to five algebraic equations while the remaining methods resulted to two or four algebraic equations. The extra work done in Perturbed SDCM is being compensated for, in terms of the accuracy obtained.

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